On Optimal Beyond-Planar Graphs

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Abstract

A graph is *beyond-planar* if it can be drawn in the plane with a specific restriction on crossings. Several types of beyond-planar graphs have been investigated, such as k-planar graphs, where every edge is crossed at most k times, and RAC graphs, where edges can cross only at a right angle in a straight-line drawing. A graph is *optimal* if the number of edges coincides with the density for its type. Optimal graphs are special and are known only for some types of beyond-planar graphs, including 1-planar, 2-planar, and RAC graphs.

For all types of beyond-planar graphs for which optimal graphs are known, we compute the range for optimal graphs, establish combinatorial properties, and show that every graph is a topological minor of an optimal graph. The minor property is well-known for general beyond-planar graphs.

Keywords and phrases topological graphs, graph drawing, beyond-planar graphs, density, minors

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1 Introduction

Graphs are often defined by particular properties of a drawing. The planar graphs, in which edge crossings are excluded, are the most prominent example. Every n-vertex planar graph has at most $3n-6$ edges. The bound is tight for triangulated planar graphs, such that they are the *optimal* planar graphs. The planar graphs have been characterized by the forbidden minors K_5 and $K_{3,3}$ [30]. The minors cannot be a subdivision of a planar graph [41], that is a *topological minor*, nor obtained by edge contraction [49]. It is well-known that every topological minor is a minor, but not conversely, see [30]. In fact, the complete graph K_5 cannot be a topological minor of any graph of degree at most three.

There has been a recent interest in *beyond-planar graphs* [29, 36, 40], which are defined by drawings with specific restrictions on crossings. These graphs are a natural generalization of the planar graphs. Their study in graph theory, graph algorithms, graph drawing, and computational geometry can provide significant insights for the design of effective methods to visualize real-world networks, that are non-planar, in general.

A graph is k*-planar* [43] if it has a drawing in the plane such that each edge is crossed by at most k edges. In particular, it is 1-planar if each edge is crossed at most once [44]. A 1-planar drawing is IC-*planar* (independent crossing) if every vertex is incident to at most one crossed edge [6], and NIC-*planar* (near-independent crossing) if two pairs of crossing edges share at most one vertex [50]. A drawing is *1-gap planar* if there is a gap in at least one edge that is part of a crossing of two edges and each edge has at most one gap [10]. A drawing is *fan-crossing free* [26] if each edge is only crossed by independent edges, i.e., they have distinct vertices, and *fan-crossing* [20] if the crossing edges have a common vertex, i.e., they form a fan. A fan-crossing drawing is *fan-planar* if each edge is crossed only from one side [38, 39]. For 1-*fan-bundle graphs* [7], edges incident to a vertex are first bundled and a bundle can be crossed at most once by another bundle. An edge can only be bundled at one of its vertices, that is on one side. At last, a drawing is *quasi-planar* [4] if three edges do not cross mutually. These properties are topological. They hold for *embeddings*, which are equivalence classes of topologically equivalent drawings. Right angle crossing (RAC) is a geometric property, in which the edges are drawn straight-line and may cross at a right angle [28]. A k-bend RAC drawing is a polyline drawing such that every edge is drawn with at most k bends and there is a right angle at crossings. Particular beyond-planar graphs can be defined by first order logic formulas [14] and in terms of an avoidance of (natural and radial) grids [2].

Beyond-planar graphs have been studied with different intensity and depth. In particular, the *density*, which is an upper bound on the number of edges of n-vertex graphs, the size of the largest complete (bipartite) graph [8], and inclusion relations have been investigated [29, 36]. The linear density is a typical property of beyond-planar graphs, see Table 1. Small complete graphs K_k with $k \leq 11$ distinguish some types, see [8, 21]. Inclusions are canonical, in general, such that a restriction on drawings implies a proper inclusion for the graph classes [29]. In particular, every RAC drawing is both fan-crossing free and quasi-planar, and the RAC graphs are a proper subclass of the fan-crossing free and the quasi-planar graphs, since the latter admit denser graphs. There are mutual incomparabilities, e.g., between RAC, fan-crossing and 2-planar graphs [21]. Also RAC graphs are incomparable with each of 1-planar graphs [32], NIC-planar graphs [9] and k-planar graphs for every fixed k [21]. However, every IC-planar graph is both 1-planar and RAC [22]. Obviously, every 2-planar graph is 1-gap planar, but not conversely [10], and every fan-planar graph is fan-crossing, but not conversely, where for every fan-crossing graph there is a fan-planar graph on the same set of vertices and with the same number of edges [20].

The situation is simpler for optimal graphs. A graph is *optimal* if its number of edges meets the established bound on the density of graphs of its type. Hence, the density is tight for values of n for which there are optimal n-vertex graphs. The term optimal has been introduced by Bodendiek et al. [12] who have studied optimal 1-planar graphs [13]. At other places *extreme* or *maximally dense* is used.

Optimal graphs are on top of an augmentation of graphs by additional edges, such that the defining property of the graphs is not violated, that is the type is preserved. Augmented graphs can often be handled more easily. A drawing of a graph is (*planar-maximal*) *maximal* if no further (uncrossed) edge can be added without violation [5]. A graph G is *maximal* for some type τ if $G + e$ is not a τ-graph for any edge e that is added to G. Note that there are *densest* and *sparsest* graphs, which are maximal graphs with the maximum and minimum number of edges among all n-vertex graphs in their type [9, 23].

We are aware of optimal graphs for the following types of beyond-planar graphs: 1-planar, 2-planar, IC-planar, NIC-planar, 1-fan-bundle and RAC graphs, see Proposition 1 and Table 1. There are more types of beyond-planar graphs with a density of $4n - 8$ and $5n - 10$, respectively, namely fan-crossing free [26] and grid crossing graphs [14] as well as fan-planar [38], fan-crossing [20], 1-gap planar [10] and 5-map graphs [18], where 5-map graphs are simultaneously 2-planar and fan-crossing (see also [29] for definitions), so that there are optimal graphs for these types, too.

Optimal graphs are yet unknown for types of beyond-planar graphs with a density above $5n - 10$. The known bounds of $5.5n - 11$ for 3-planar [11] and $6.5n - 20$ for quasi-planar graphs [3] are tight up to a constant. Similarly, there is a constant gap for graphs with geometric thickness two (doubly linear) [37], and for bar-visibility and rectangle visibility graphs [37]. There are larger gaps for k-planar graphs with $k \geq 4$ [1, 43], 2- and 3-bend RAC graphs [28], and graphs avoiding special grids [2, 42].

In general, the recognition of beyond-planar graphs is NP-hard [29]. However, optimal 1-planar [17], optimal 2-planar [34], and optimal NIC-planar graphs [9] can be recognized in linear time and optimal IC-planar graphs in cubic time [16]. The recognition problem for optimal RAC, optimal 1-gap planar, optimal fan-crossing and optimal 1-fan-bundle graphs is open. Since an optimal RAC graph is 1-planar and triangulated, the pairs of crossing edges can be computed in cubic time [19], so that it remains to determine whether or not all pairs of crossing edges can cross at a right angle in a straight-line drawing.

Our contribution: In this paper, we consider optimal graphs of the aforementioned types τ of beyond-planar graphs. We study combinatorial properties and compute the range for optimal graphs, which was only known partially in some cases. We show that every graph has a subdivision that is a subgraph of an optimal τ-graph, whereas there are optimal τ-graphs that contain K_{42} as a minor but not as a topological minor.

The paper is organized as follows: We introduce basic notions on beyond-planar graphs in the next section and recall some properties of such graphs. We study combinatorial properties of optimal graphs in Section 3 and minors in Section 4 and we conclude in Section 5.

2 Preliminaries

We consider graphs that are *simple* both in a graph theoretic and in a topological sense. Thus there are no multi-edges or loops, adjacent edges do not cross, and two edges cross at most once in a drawing. A graph $G = (V, E)$ consists of sets of n vertices and m edges. We assume that it is defined by a drawing $\Gamma(G)$ in the plane. The *planar skeleton* of G (or $\Gamma(G)$) is the subgraph induced by the uncrossed edges.

A crossed quadrangle, X-*quadrangle* for short, is a planar quadrangle with a pair of crossing edges in its interior. There are no other vertices or edges in the interior, as opposed to [23]. At several other places, the term kite has been used. Similarly, an X-*pentagon* consists of a pentagon of five uncrossed edges and only a pentagram of five crossing edges in its interior. These drawings of K_4 and K_5 have been used for optimal 1- and 2-planar graphs [11, 13, 43]. We say that a vertex (edge) *is in a triangle* if it is in the boundary of the triangle, and in an X-quadrangle if is a part of the X-quadrangle.

We consider beyond-planar graphs of *type* τ, where τ ranges over the set of beyond-planar graphs for which optimal graphs are known: IC-planar, NIC-planar, 1-planar, 2-planar, 1-fan-bundle and right angle crossing (RAC) graphs, as well as fan-crossing free, grid crossing, fan-crossing, fan-planar, 1-gap-planar and 5-map graphs.

For convenience, we do not distinguish between a graph, a drawing or an embedding, and we assume that a graph and its drawing or embedding are always of the same type.

Next we summarize related work on beyond-planar graphs of type τ.

▶ **Proposition 1.** **(i)** *An n-vertex graph is optimal 1-planar if it has $4n - 8$ edges [13]. A graph G is optimal 1-planar if and only if the planar skeleton is a 3-connected quadrangulation [46], such that G is obtained by inserting a pair of crossing edges in each quadrangle. There are optimal 1-planar graphs if and only if $n = 8$ and $n \geq 10$ [13]. The number of optimal 1-planar graphs is known for $n \leq 36$ [25]. Optimal 1-planar graphs have a unique embedding, except for extended wheel graphs $XW_{2k}, k \geq 3$, which have two embeddings for $k \geq 4$ [45] and eight for $k = 3$ [47]. The embeddings are unique* *up to graph isomorphism [47]. Optimal 1-planar graphs can be recognized in linear time [17].*

(ii) *Every optimal fan-crossing free graph is 1-planar [26], and thus optimal 1-planar.*

(iii) *An n-vertex graph is optimal IC-planar if it has $\frac{13}{4}n-6$ edges [51]. There are optimal IC-planar graphs if and only if $n=4k$ and $k \geq 2$ [51]. Optimal IC-planar graphs can be recognized in cubic time [16].*

(iv) *An n-vertex graph is optimal NIC-planar if it has $\frac{18}{5}(n-2)$ edges [50]. There are optimal NIC-planar graphs if and only if n=5k+2 for $k \geq 2$ [9, 27]. Optimal NIC-planar graphs have a unique NIC-planar embedding and can be recognized in linear time [9].*

(v) *An n-vertex graph is optimal 2-planar if it has $5n-10$ edges [43]. A graph G is optimal 2-planar if and only if the planar skeleton is a 3-connected pentangulation [11], such that G is obtained by substituting each pentagram by an X-pentagram, that is adding a pentagram of crossed edges in each pentagon. The number of optimal 2-planar graphs is known for $n \leq 36$ [35]. Every optimal 2-planar graph is an optimal fan-crossing [15], fan-planar [38], 1-gap planar[10], and 5-map graph [18]. There are optimal 2-planar graphs for every $n \geq 50$ with $n = 2 \bmod 3$ [43]. Optimal 2-planar graphs can be recognized in linear time [34].*

(vi) *There are optimal 1-gap planar graphs [10] and optimal fan-planar (fan-crossing) graphs [38] for every $n \geq 20$.*

(vii) *An n-vertex graph is optimal 1-fan-bundle if it has $\frac{13}{3}(n-2)$ edges [7]. Every optimal 1-fan bundle graph consists of a planar pentangulation, such that four crossing edges are inserted in each pentagon (without creating a multi-edge). There are optimal 1-fan bundle graphs if $n = 2 \bmod 3$ for properly chosen values of n [7].*

(viii) *An n-vertex graph is optimal RAC if it has $4n-10$ edges [28]. Every optimal RAC graph is 1-planar [32]. The outer face of a drawing is a triangle. There are optimal RAC graphs if $n = 3k-5, k \geq 2$ [28].*

type	density	range of optimal graphs
1-planar	$4n-8$ [13]	$n=8$ and $n \geq 10$ [13]
IC-planar	$\frac{13}{4}n-6$ [51]	$n=4k, k \geq 2$ [51]
NIC-planar	$\frac{18}{5}(n-2)$ [50]	$n=5k+2, k \geq 2$ [9, 27]
2-planar	$5n-10$ [43]	$n=20$ and $n=3k+2, k \geq 8$ (*)
1-gap planar	$5n-10$ [10]	$n \geq 20$ or $n=10,11,12,15,16,17$ (*)
1-fan-bundle	$\frac{18}{5}(n-2)$ [7]	$n=3k+2, k \geq 2$ (*)
fan-crossing	$5n-10$ [20, 39]	$n \geq 20$ [39]
RAC	$4n-10$ [28]	$n \geq 4$ (*)

Table 1 Some types of beyond-planar graphs, their density and optimal graphs. A (*) indicates an extension of the range in this work.

3 Combinatorial Properties

We now improve some results from Proposition 1.

▶ **Lemma 2.** *An IC-planar drawing of an optimal IC-planar graph consists of triangles and X-quadrangles. Every vertex is in a triangle and in exactly one X-quadrangle. X-quadrangles are vertex disjoint. Every uncrossed edge is in a triangle. There exist optimal IC-planar graphs with exponentially many IC-planar embeddings.*

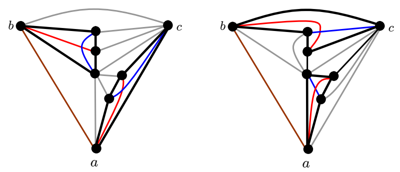


Figure 1 An IC-planar graph with two IC-planar embeddings (from [16]).

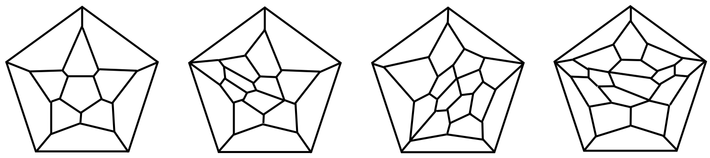


Figure 2 The smallest pentangulations with 20, 26, 29 and 32 vertices. Graph (a) is known as dodecahedron graph.

Proof. X-quadrangles are vertex disjoint by IC-planarity. An optimal IC-planar graph has $\frac{n}{4}$ X-quadrangles, such that every vertex is in an X-quadrangle. Also every uncrossed edge must be in some triangle, since X-quadrangles do not share vertices or edges. The graph in Fig. 1 is optimal IC-planar and has two IC-planar embeddings. By taking k copies and a subsequent triangulation for optimality, there is an IC-planar graph that has 2^k many embeddings, where $k = n/8$. ◀

▶ **Lemma 3.** *A NIC-planar drawing of an optimal IC-planar graph consists of triangles and X-quadrangles, such that every edge is in an X-quadrangle and every uncrossed edge is in a triangle.*

Proof. An optimal NIC-planar graph has $\frac{3}{5}(n-2)$ X-quadrangles and $\frac{18}{5}(n-2)$ edges. Since two X-quadrangles do not share an edge in a NIC-planar embedding, every edge is in an X-quadrangle and every uncrossed edge is a triangle. Hence, there is an X-quadrangle on one side of each uncrossed edge and a triangle on the other side. Note that the claim can also be obtained from Corollary 4 in [9]. ◀

The full range for optimal 1-planar, IC-planar and NIC-planar graphs has been discovered before, but only partially for 2-planar, 1-fan-bundle, and RAC graphs, as stated in Proposition 1.

▶ **Theorem 4.** *There are optimal 2-planar graphs if and only if $n = 20$ or $n \geq 26$ and $n = 2$ mod 3.*

Proof. There are 3-connected and even 5-connected 5-regular planar graphs with n faces if and only if $n = 20$ or $n \geq 26$ and $n = 2$ mod 3, as shown by Hasheminezhad et al. [35]. The dual is a 3-connected pentangulation, which is turned into an optimal 2-planar if and only if each pentagon is substituted by an X-pentagon, that is, it is filled by a pentagram, as shown in [11]. The smallest pentangulations are shown in Fig. 2. At other places [10, 11, 38, 43] the dodecahedron graph, shown in Fig. 2a, has been used, where recursively the dodecahedron graph is substituted in a face. ◀

Note that there is no optimal 1-planar graph with seven or nine vertices [13] and no optimal 2-planar graph with $21, \ldots, 25$ vertices. Moreover, the number of optimal 1-planar

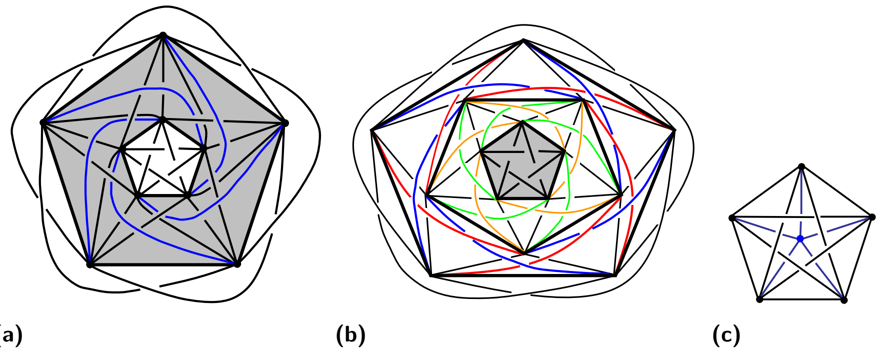


Figure 3 Optimal 1-gap planar graphs with (a) $n = 10$ and (b) $n = 15$ vertices and (c) the addition of a vertex to augment K_5 to K_6 for 1-gap planar drawings.

and optimal 2-planar graph is known for $n \leq 36$ from the number of 3-connected quadrangulations [25] and 3-connected 5-regular graphs [35] graphs, namely 3915683667721 and 16166596 for $n = 36$.

Clearly, every optimal 2-planar graph is optimal 1-gap planar, optimal fan-crossing, and optimal fan-planar, since the density is $5n - 10$ and the latter extend the 2-planar graphs. The graphs are simultaneously 2-planar and fan-crossing, so that they are optimal 5-map graphs [18]. Hence, the sets of optimal 2-planar and optimal 5-map graphs coincide, although there are 2-planar graphs that are not 5-map graphs. Moreover, there are further optimal 1-gap planar, fan-planar and fan-crossing graphs, since the restriction to $n = 2 \bmod 3$ can be dropped by the addition of a vertex in a pentagon [10, 38], see Fig. 3c. Surprisingly, there are small optimal 1-gap planar graphs.

▶ **Lemma 5.** *There are optimal 1-gap planar graphs for every* $n \geq 20$ *and for* $n = 10, 11, 12, 15, 16, 17$*, but not for* $n \leq 8$.

Proof. Optimal 1-gap planar graphs can be constructed as before for 2-planar graphs in Theorem 6 using 5-regular planar graphs for $n \geq 20$ and $n = 2 \bmod 3$ and filling each face such that there is a 5-clique including the boundary of the face. In general, the dodecahedron graph has been used, where large graphs are obtained by repeatedly substituting a pentagon by the dodecahedron. Graphs with $n \neq 5k$ vertices for $k \geq 4$ are obtained by the addition of a vertex in the interior of a pentagram, such that there is a 6-clique, see Fig. 3c.

We introduce another substitution scheme and draw a bipartite graph with 20 edges between two pentagons, such that the drawing is 1-gap planar, see Fig. 3a. So we obtain optimal 1-gap planar graphs with 10 and 15 vertices, see Fig. 3b. Optimal 1-gap planar graphs with $5k + 1$ and $5k + 2$ edges for $k \geq 2$ are obtained by substituting the inner and/or outer K_5 by K_6, as shown in Fig. 3c.

Clearly, there are no optimal 1-gap planar graphs for $n \leq 8$, since such graphs have too few edges. ◀

Hence, optimal 1-gap planar graphs are not known for $n = 9, 13, 14, 18$ and 19, and we conjecture that there are no such graphs.

▶ **Theorem 6.** *There are optimal 1-fan-bundle graphs exactly for every* $n \geq 8$ *if* $n = 2 \bmod 3$.

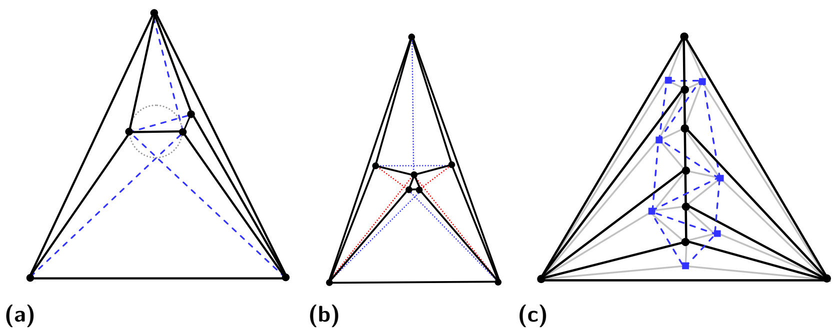


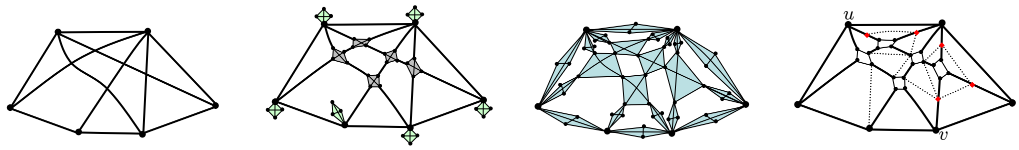
Figure 4 RAC drawings of optimal RAC graphs with (a) six and (b) eight vertices, and (c) a vertex-face graph.

Proof. Since optimal 1-fan-bundle graphs have $\frac{13}{3}(n-2)$ edges, there are optimal graphs only for $n = 3k+2$ by integrality. The 5-clique minus one edge has only nine edges, so that it is not optimal 1-fan-bundle. Optimal graphs for $n = 3k+2$ and $k \geq 2$ can be obtained from a planar pentangulation and the insertion of four edges in each pentagon. A pentangulation may have vertices of degree two, as opposed to the previous case for 2-planar graphs. Then the neighbors of a degree two vertex are connected by an edge in only one of the adjacent faces. The smallest optimal 1-fan-bundle graph is shown in Fig. 6b. By induction, remove the crossed edges in a pentagon P, add three vertices as Steiner points, partition P into three pentagons, such that there are two vertices of degree two, and insert four crossed edges in each face, such that there is no multi-edge, see Fig. 6b. So we obtain optimal graphs for every $n = 3k+2, k \geq 2$. ◀

A *primal-dual graph* of a 3-connected planar graph G is obtained by the simultaneous drawing of G and its dual G^*, from which the dual vertex for the outer face has been removed. Every primal edge of G is crossed by the dual edge between the faces on either side. Moreover, every dual vertex for a face of G is adjacent to every primal vertex in the boundary of the face, see Fig. 4c. A primal-dual graph is 1-planar. Primal-dual graphs (including the outer face) have been used by Ringel [44] in his early study of 1-planar graphs. Brightwell and Scheinerman [24] have shown that primal-dual graphs admit a RAC drawing, see also [31, 33].

▶ **Theorem 7.** *There are optimal RAC graphs for every $n \geq 4$.*

Proof. Clearly, K_4 and K_5 are optimal RAC graphs. Optimal RAC graphs with six and eight vertices are displayed in Fig. 4. In general, an optimal RAC graph can be constructed from a primal-dual graph, as observed by Didimo et al [28]. In particular, optimal RAC graphs with $n \geq 9$ can be constructed a follows. Add $k \geq 2$ vertices $v_1, \ldots, v_k$ in the interior of an outer triangle $\Delta(a,b,c)$ with c on top. Add edges cv_k and v_iv_{i+1} for $i = 1, \ldots, k-1$, such that there is a path. Next add edges such that the so obtained graph is 3-connected and inner faces are triangles or quadrangles. For example, add edges av_1, bv_1 and av_i and bv_j, where $2 \leq i, j \leq k$ and i is odd and j is even, such that there are $k-1$ quadrangles in an n-vertex graph with $n = k+3$. Then there are three inner triangles and $n-4$ quadrangles. The primal-dual graph has $2n-1$ vertices and $8n-14$ edges, $2n-2$ of which are in the primal graph, $2n-5$ are in the dual graph and $4(n-4)+9$ edges are added for the primal-dual combination, and it is 1-planar [44]. One more vertex and four more edges are obtained if a quadrangle of the primal graph is triangulated. The so obtained primal-dual graph is 3-connected and admits a RAC drawing [24], so that there is an optimal RAC graph for



(a) a drawn graph **(b)** an IC-planar host **(c)** a NIC-planar host **(d)** a 1-planar host

Figure 5 Illustration for the proof of Theorem 8. Each face of the host must be filled (b) by a triangulation, (c) by gadgets from Fig. 6a and (c) by a pair of crossing edges. Vertices u and v have degree four and dots are propagated from u to v in cw-order.

every $n \geq 9$. The one for $n = 7$ is obtained in the same way from a planar drawing of K_4, see [28]. ◀

4 Minors

It is well-known that for any graph there exists a 1-planar subdivision, that is any graph is a topological minor of a 1-planar graph. This fact has been improved to IC-planar graphs and to upper bounds on the number of subdivisions [21]. In particular, any graph has a 3-subdivision that is RAC [28], a 2-subdivision that is fan-crossing, and a 1-subdivision that is quasi-planar [21]. We consider minors in optimal graphs.

▶ **Theorem 8.** *For any graph G there is an optimal beyond-planar graph H of type τ, where τ is IC-planar, NIC-planar, 1-planar, 2-planar, 1-fan-bundle and RAC, respectively, such that G is a topological minor of H.*

Proof. Consider a drawing of G and treat it as a planar graph Γ, such that a crossing point of two edges of G is a new vertex of degree four. We construct a host graph H by placing a "gadget" at each crossing point, such that the crossing happens in the gadget. Thereafter, the intermediate graph is augmented for optimality. The gadget is an X-quadrangle if the type is 1-planar, IC-planar, NIC-planar, and RAC, respectively, and an X-pentagon or the dodecahedron graph with crossing edges in each inner face for 2-planar and 1-fan-bundle graphs.

First, for IC-planar graphs, we substitute each vertex v of G by an X-quadrangle, such that v is one vertex of the X-quadrangle and there are three new vertices not in G. Then replace each crossing by an X-quadrangle, see Fig. 5b. Every edge of G is partitioned into segments, which are uncrossed pieces in the drawing. These segments are inherited by H such that a segment connects vertices in two X-quadrangles. There are no further vertices in H, so that every vertex is in an X-quadrangle. Two X-quadrangles are vertex disjoint, which guaranties IC-planarity. By Lemma 2, optimality is obtained by a triangulation. Clearly, for every edge of G there is a path in H such that two such paths are vertex disjoint.

Similarly, for NIC-planar graphs, we replace each crossing by an X-quadrangle such that the X-quadrangles for two consecutive crossings along an edge share a vertex, see Fig. 5c. In other words, segments are contracted. An uncrossed edge and the first (last) segment of a crossed edge incident to a vertex of G is replaced by an X-quadrangle with the segment as a diagonal. Thereby, every edge of G is subdivided, such that two paths for edges of G are vertex disjoint in H. It remains to construct an optimal NIC-planar graph by filling the remaining faces. In addition, we wish to keep the degree low. So far, the boundary of each face consists of edges from X-quadrangles. If there is a triangle, we are done. The gadget from Fig. 6a is inserted in a quadrangle. Larger faces are partitioned using X-quadrangles and triangles and the gadget for quadrangles, such that each edge of H is in an X-quadrangle and in a triangle. Then H is an optimal NIC-planar graph by Lemma 3.

The planar skeleton of an optimal 1-planar graph is a 3-connected planar quadrangulation [46]. It is obtained from G as follows. First, assume that G is 3-connected, otherwise, add edges to obtain 3-connectivity, such that the degree of a vertex increases at most by three. Next, draw G such that all vertices are in the outer face and the crossings in the interior. Let $v_1, \ldots, v_n$ be the vertices of G in cyclic order in the so obtained drawing Γ_1. Assume that there is an uncrossed edge $v_i v_{i+1}$ for $i = 1, \ldots, n$ with $v_1 = v_{n+1}$ between two consecutive vertices in Γ_1, where it is added if it was missing and is rerouted such that it is uncrossed, otherwise. Next, assume that the number of vertices of even degree is even. Otherwise, add a triangle to Γ_1, such that its vertices are in the outer face and any two of them are not consecutive. Each added vertex is connected to its neighbors in the outer face, such that it has degree four. Let Γ_2 be the so obtained drawing. Its outer face is bounded by a cycle of uncrossed edges of even length, since both the number of vertices with odd and with even degree is even.

Construct a planar drawing Γ_3 from Γ_2 by substituting each crossing by a quadrangle. Thereby, we obtain a 3-connected planar graph. Next, consider all peripheral faces of Γ_3, which are faces with a vertex of G in the boundary other than the outer face. By assumption, a peripheral face has one or two vertices of G in its boundary and an even number of vertices from the substitution of crossings by quadrangles. For the subsequent quadrangulation, edges of Γ_3 can be subdivided if they are incident to an outer vertex. Traverse the vertices in the outer face of Γ_3 in cyclic order. Let v be a vertex of degree d in the outer face of Γ_3. Then the first and the last peripheral face incident to v have an even size and the size is odd for the peripheral faces in between. Let $e_1, \ldots, e_d$ be the edges at vertex v in cyclic order, where e_1 and e_d are uncrossed edges in the outer face. Subdivide edge e_{2i+1} for $i = 1, \ldots, \lceil \frac{d-3}{2} \rceil$ if v does not inherit a so called "dot" from its predecessor, and subdivide edge e_{2i} for $i = 1, \ldots, \lfloor \frac{d-1}{2} \rfloor$, otherwise. Vertex v_i propagates a dot to v_{i+1} if its last but one edge e_{d-1} is subdivided. Otherwise, the dot is consumed. Let Γ_4 be the so obtained planar drawing, see Fig. 5d.

Now quadrangulate each face of size at least six of Γ_4 by adding uncrossed edges, such that each vertex of degree two, that is a subdivision vertex, is incident to at least one edge that is added for a quadrangulation. We claim that the so obtained drawing Γ_5 is a 3-connected planar quadrangulation. To see this, first observe that edges of Γ_5 do not cross by construction. Second, the graph from Γ_5 is 3-connected, since G is 3-connected and connectivity is preserved by the construction. In particular, if p is subdivision vertex on edge uv, then p is incident to another vertex w by the quadrangulation. Now there are three vertex disjoint path between p and any other vertex of Γ_5 that pass through u, v and w. Finally, the size of each face of Γ_4 is even. This holds for the outer face, since G has an even number of vertices, and for any face, that is not peripheral, since each crossing point in a face of Γ_3 is replaced by two vertices from a quadrangle. Also the size of each peripheral face of Γ_5 is even. Therefore, consider the peripheral faces $f_1, \ldots, f_{d-1}$ incident to vertex v_i, where v_i has degree $d \geq 3$. If v_i does not inherit a dot from its predecessor, then the size of f_1 is even. By a subdivision of edge e_{2i+1} for $i = 1, \ldots, \lceil \frac{d-3}{2} \rceil$, the size of face f_j for $j = 2, \ldots, d-2$ is increased by one, so that it is even. If d is even, then also the size of f_{d-1} is increased by one. Then a dot is propagated, such that the size of f_{d-1} is increased again as the first face of v_{i+1}. The case in which v_i inherits a dot is similar. Since the number of vertices of even (and odd) degree is even, all created dots are consumed. Hence, the size of a face of Γ_4 is even. Thus it can be quadrangulated, so that Γ_5 is a planar quadrangulation.

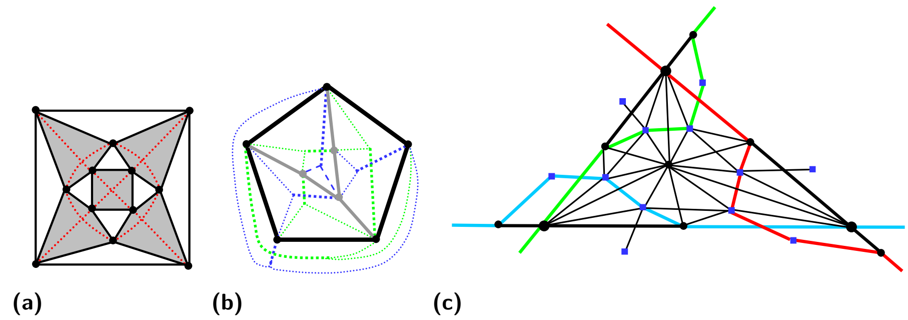


Figure 6 Illustration for the proof of Theorem 8. (a) A NIC-planar graph in a quadrangle. (b) The smallest optimal 1-fan-bundle graph with three Steiner points for a pentangulation of a pentagon. (c) A bypass for three pairwise crossing edges using a primal-dual graph and auxiliary vertices. are colored red, blue and green.

Finally, add a pair of crossed edges in each quadrangle. Thereby, we obtain a optimal 1-planar H that contains a subdivision of G as a subgraph, since there is an X-quadrangle for each crossing of edges of G.

For 1-planarity, the size of the faces of the drawing of G is even if the crossing points are replaced by X-quadrangles and the first (last) segment between a vertex and a crossing point is subdivided. Then the faces can be partitioned into quadrangles, which are augmented to X-quadrangles, see [48]. The planar skeleton is 3-connected [5], so that H is optimal 1-planar.

For 2-planarity, the drawing of G must be transformed into a 3-connected pentangulation. Therefore, we first replace every crossing point in the drawing of G by the dodecahedron graph, as shown in Fig. 2a, such that the segment of an edge between two crossing points is attached to two outer vertices on opposite sides of the outer pentagon. Alternatively, two vertices of adjacent pentagons coincide, as before in the NIC-planar case. Clearly, the so obtained graph is planar. If every inner pentagram is later filled by a pentagram, then two crossing edges can be routed internally such that their subdivisions are vertex disjoint. Hence, there is a subdivision of G. Next, large faces of size at least six are partitioned into pentagrams, triangles and quadrangles such that there are no vertices of degree two and the intermediate graph is 3-connected. If there is a quadrangle, then insert the dodecahedron graph in its interior and partition the region in between into three more pentagons. Similarly, the region between a triangle and an inserted dodecahedron graph is partitioned into two pentagons and a quadrangle, which is partitioned into pentagons as described before. Finally, all pentagons are filled by pentagrams such that there are only X-pentagons. This does not create a multi-edge, since every vertex of the planar skeleton has degree at least three, and there is no separation pair, so that the pentangulation is 3-connected. Two crossing edges from the drawing of G are replaced by two short paths that can be routed internally through the X-pentagrams such that the paths are vertex disjoint. Hence, there is an optimal 2-planar graph that contains a subdivision of G as a subgraph.

For 1-fan-bundle graphs, we proceed as before, but finally fill the pentagrams by four edges, such that an edge crossing in G is transferred to an edge crossing inside a pentagon.

For RAC graphs, suppose that G is drawn with straight-line segments. Treat the drawing as a planar graph and triangulate it. Then subdivide each edge, place a new vertex in each face (except for the outer face) and connect the inserted vertex with the six vertices in the boundary of the face such that there is a re-triangulation. Now construct the primal-dual graph H, which is 1-planar and an optimal RAC graph, since it admits a RAC drawing as shown by Brightwell and Scheinerman [24]. It remains to consider paths for edges of G. Every edge of G has a subdivision in H which uses the segments from the drawing. Then two paths meet in the crossing point for their edges. This collision is circumvented using a bypass. Suppose that edges e, f and g cross each other such that there is a triangle in the drawing of G with crossing points u, v and w, see Fig. 6c. All other cases are similar. Then e can bypass u and pass through w, f can pass through u and bypass v, and h can bypass w and pass through v. For edge e, the bypath begins at the subdivision point just before u, it goes through the (vertices for the) faces next to u, and ends at the subdivision point on e after u. Thereby it crosses the path for edge g. Hence, there are vertex disjoint paths in the vertex face graph H for the edges of G, such that a subdivision of G is a subgraph of H, that is G is a topological minor of H. ◀

▶ **Corollary 9.** *Any graph is a topological minor of an optimal graph for each of the following types: fan-crossing free, grid-crossing, fan-crossing, fan-planar, 1-gap-planar, 4-map, 5-map.*

Proof. We can use the constructions for 1-planar and 2-planar graphs from (the proof of) Theorem 10, since optimal fan-crossing free (grid crossing) graphs are optimal 1-planar [26], and every 4-map graph is a 3-connected triangulated 1-planar graph. Similarly, the crossed dodecahedron graph is a 5-map graph [18] and simultaneously 2-planar and fan-crossing, such that the construction for 2-planar graphs can be used for all types of graphs with a density of $5n - 10$. ◀

Finally, we distinguish topological minors from minors in optimal beyond-planar graphs. Similar to the case of K_5 and graphs of degree at most three, we wish to keep the degree of graphs low if they contain a (large) clique as a minor. The degree is determined by the gadgets for edge crossings and the filling of faces towards optimality. The degree of the minor can be decreased to three.

A 3-*regularization* transforms a graph into a graph of degree three by a local operation on vertices. Examples are the *node-to-circle expansion*, which expands every vertex of degree d into a circle of d vertices of degree three [21], or the expansion into a binary tree with d leaves. Each vertex on the circle (leaf) inherits one incident edge. 3-regularization preserves minors, such that G is a minor of a 3-regularization $\eta(H)$ if G is a minor of H. However, 3-regularization does not preserve topological minors, since the obtained graphs have degree at most three, and therefore exclude any graph with a vertex of degree at least four as a topological minor. In consequence, K_5 is not a topological minor of any 3-regularization.

▶ **Theorem 10.** *For every type τ of beyond-planar graphs as above, there is a constant d_τ and an optimal τ-graph H, such that the complete graph K_k with $k = d_\tau$ is a minor but not a topological minor of H. In the above cases, we have $d_\tau \leq 42$.*

Proof. For every type τ, there is an optimal τ-graph H, such that the 3-regularization of K_d is a (topological) minor of H by Theorem 8. Recall the construction of H from the proof of Theorem 8 and try to keep the degree low.

For IC-planar graphs, there is a triangulation of faces that increases the degree of each vertex by at most two, and there are at most two faces for each vertex, since there are gadgets for the crossing points. Hence, H has degree at most eight, so that there are optimal IC-planar graphs that contain $\eta(K_{10})$ as a minor, but not as a topological minor.

Similarly, the degree of H can be kept as low as 24 for NIC-planar graphs, and 14 for 1-planar graphs, since X-quadrangles and gadgets are used for a partition of large faces.

For RAC graphs, we first subdivide each edge of the planar graph from the drawing and create triangles with two subdivision points and a vertex or a crossing point from the drawing such that each crossing point is surrounded by four triangles. Then we triangulate the remaining faces such that the degree of each subdivision point is increased at most by two from the face on either side. The intermediate graph is a triangulated planar graph of degree at most ten. In the next step, we use a primal-dual graph for each triangle such that graph H has degree at most 40.

For 2-planar and 1-fan-bundle graphs, there is a pentagon around each vertex and each crossing point in the drawing of G. Faces can be filled such that at most most four dodecahedron graphs meet in a point. Thereby, graph H has degree at most 12.

In any case, graph H does contain a topological minor of degree at least $d+1$, so that K_{d+2} is a minor but not a topological minor of H. ◀

5 Conclusion

In this work, we study optimal graphs for some important types of beyond-planar graphs. We compute and extend the range for such graphs and show that optimal graphs contain any graph as a (topological) minor.

Open problems include tight bounds on the density of further beyond-planar graphs including 3-planar and 1-bend RAC graphs, the characterization of optimal fan-crossing and optimal 1-gap planar graphs, and the recognition problem for optimal RAC graphs.

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