Morphing Planar Graph Drawings Through 3D

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Abstract

In this paper, we investigate crossing-free 3D morphs between planar straight-line drawings. We show that, for any two (not necessarily topologically equivalent) planar straight-line drawings of an n-vertex planar graph, there exists a crossing-free piecewise-linear 3D morph with $O(n^2)$ steps that transforms one drawing into the other. We also give some evidence why it is difficult to obtain a linear lower bound (which exists in 2D) for the number of steps of a crossing-free piecewise-linear 3D morph.

Keywords and phrases Linear morph, 3-dimensional graph drawing, morphing steps

Digital Object Identifier 10.57717/cgt.v2i1.33

Funding Research partially supported by PRIN projects no. 2022ME9Z78 “NextGRAAL” and no. 2022TS4Y3N “EXPAND”.

1 Introduction

A *morph* is a continuous transformation between two given drawings of the same graph. A morph is required to preserve specific topological and geometric properties of the input drawings. For example, if the drawings are planar and straight-line, the morph is required to preserve such properties throughout the transformation. A morphing problem often assumes that the input drawings are “topologically equivalent”, that is, they have the same “topological structure”. For example, if the input drawings are planar, they are required to have the same rotation system (i.e., the same clockwise order of the edges incident to each vertex) and the same walk bounding the outer face; this condition is obviously necessary (and, if the graph is connected, also sufficient [7, 12]) for a morph to exist between the given drawings. A *linear* morph is a morph in which each vertex moves along a straight-line segment, all vertices leave their initial positions simultaneously, move at uniform speed, and arrive at their final positions simultaneously. A *piecewise-linear* morph consists of a sequence of linear morphs, called *steps*. A recent line of research culminated in an algorithm by Alamdari et al. [3] that constructs a piecewise-linear morph with $O(n)$ steps between any two topologically equivalent planar straight-line drawings of the same n-vertex planar graph; this bound is worst-case optimal.

What can one gain by allowing the morph to use a third dimension? That is, suppose that the input drawings still lie on the plane $z = 0$, does one get "better" morphs if the intermediate drawings are allowed to live in 3D? Arseneva et al. [5] proved that this is the case, as they showed that, for any two planar straight-line drawings of an n-vertex tree, there exists a crossing-free (i.e., no two edges cross in any intermediate drawing) piecewise-linear 3D morph between them with $O(\log n)$ steps. Later, Istomina et al. [6] gave a different algorithm for the same problem. Their algorithm uses $O(\sqrt{n}\log n)$ steps, however it guarantees that any intermediate drawing of the morph lies on a 3D grid of polynomial size.

Our contribution. We prove that the use of a third dimension allows us to construct a morph between any two, possibly *topologically non-equivalent*, planar drawings. Indeed, we show that $O(n^2)$ steps always suffice for constructing a crossing-free piecewise-linear 3D morph between any two planar straight-line drawings of the same n-vertex planar graph; see Section 3. Our algorithm defines some 3D morph "operations" and applies a suitable sequence of these operations in order to modify the embedding of the first drawing into that of the second drawing. The topological effect of our operations on the drawing is similar to, although not the same as, that of the operations defined by Angelini et al. [4]. Both the operations defined by Angelini et al. and ours allow to transform an embedding of a biconnected planar graph into any other. However, while our operations are crossing-free piecewise-linear 3D morphs, we see no easy way to directly implement the operations defined by Angelini et al. (that concern topological drawings) as crossing-free piecewise-linear 3D morphs. Later, we will point out a concrete difference. We stress that the input of our algorithm consists of a pair of planar drawings in the plane $z = 0$; the algorithm cannot handle general 3D drawings as input.

We then discuss the difficulty of establishing non-trivial lower bounds for the number of steps needed to construct a crossing-free piecewise-linear 3D morph between planar straight-line drawings; see Section 4. We show that, with the help of the third dimension, one can morph, in a constant number of steps, two topologically equivalent drawings of a nested-triangle graph (see Figure 14) that are known to require a linear number of steps in any crossing-free piecewise-linear 2D morph [3].

Section 2 introduces some preliminaries and Section 5 concludes with some open problems.

2 Preliminaries

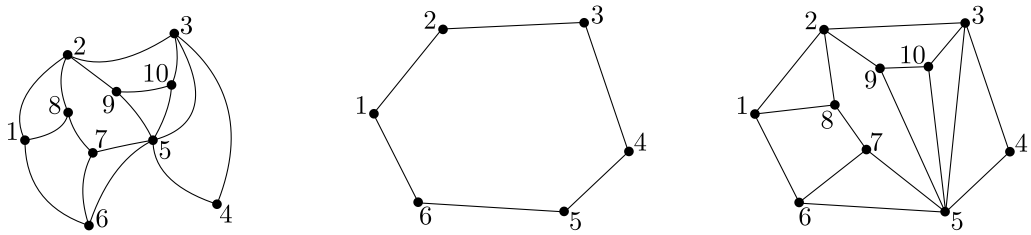
In this section, we give some definitions and preliminaries.

A *2D drawing* of a graph maps each vertex to a point in the plane and each edge to a Jordan arc in the plane between the points corresponding to its end-vertices. A *3D drawing* is defined analogously, just that points and arcs are embedded in space, rather than in the plane. A 2D drawing is *planar* if no two edges intersect except, possibly, at common end-vertices. Analogously, a 3D drawing is *crossing-free* if no two edges intersect except, possibly, at common end-vertices. A drawing is *straight-line* if each edge is represented by a straight-line segment. A planar drawing is *strictly convex* if each face is delimited by a strictly convex polygon, i.e., by a simple polygon whose internal angles are strictly less than 180°.

Throughout this paragraph, every considered graph is assumed to be connected. Two planar drawings of a graph are (*topologically*) *equivalent* if they have the same rotation system and the same clockwise order of the vertices along the boundary of the outer face.

(a) An embedding $\mathcal{E}$ **(b)** The embedding obtained as a flip of $\mathcal{E}$

Figure 1 Illustration for the flip of an embedding $\mathcal{E}$ of a graph G. In this example, $\{5,6\}$ is a separation pair (and a split pair). The three split components with respect to $\{5,6\}$ are the edge $(5,6)$, the path $(5,8,6)$, and the graph obtained from G by removing the vertex 8 and the edges $(5,6)$, $(5,8)$, and $(8,6)$.



(a) **(b)** **(c)**

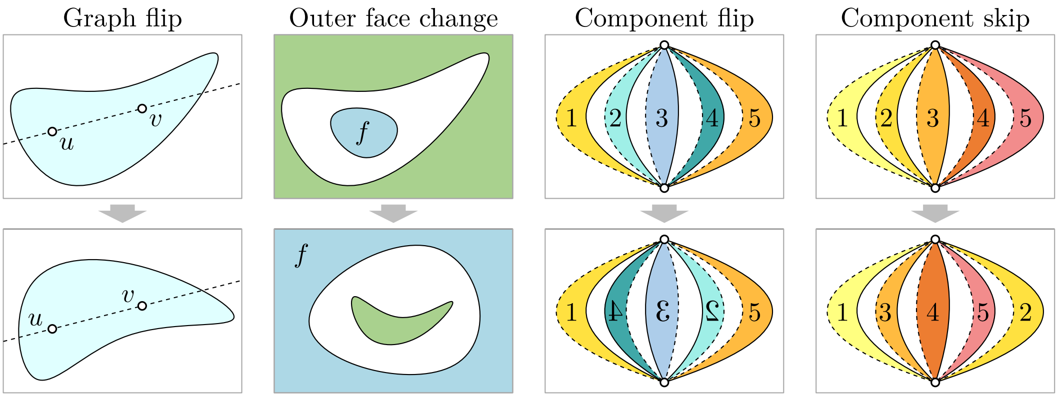
Figure 2 Illustration for Theorem 1. (a) An internally-triconnected plane graph G. (b) A strictly convex polygon P that represents the cycle delimiting the outer face of G. (c) A strictly convex drawing of G that coincides with P when restricted to the vertices and edges on the boundary of the outer face of G.

An *embedding* is an equivalence class of planar drawings of a graph. A *plane* graph is a planar graph with an embedding; when we talk about a planar drawing of a plane graph, we always assume that the embedding of the drawing is that of the plane graph. The *flip* of an embedding $\mathcal{E}$ produces an embedding in which the clockwise order of the edges incident to each vertex and the clockwise order of the vertices along the boundary of the outer face are the opposite of the ones in $\mathcal{E}$; see Figure 1.

A graph is *biconnected* (*triconnected*) if the removal of any vertex (resp. of any two vertices) leaves the graph connected. A pair of vertices of a biconnected graph G is a *separation pair* if its removal disconnects G. A *split pair* of G is a separation pair or a pair of adjacent vertices. A *split component* of G with respect to a split pair $\{u,v\}$ is the edge (u,v) or a maximal subgraph G_{uv} of G such that $\{u,v\}$ is not a split pair of G_{uv}; see again Figure 1. A biconnected plane graph G is *internally-triconnected* if every separation pair $\{u,v\}$ is such that: (i) u and v are incident to the outer face of G; (ii) every split component with respect to $\{u,v\}$ that is different from the edge (u,v) contains edges incident to the outer face of G.

Throughout the paper, we use some results from the literature, which we state here for the reader's convenience. We start with a theorem on the existence of strictly convex drawings with prescribed outer face; see also Figure 2.

▶ **Theorem 1** (Hong and Nagamochi [10]; Tutte [13]). *Let G be an internally-triconnected plane graph, and let P be a strictly convex polygon representing the cycle delimiting the outer face of G. It is possible to construct a strictly convex drawing of G that coincides with P when restricted to the vertices and edges on the boundary of the outer face of G.*



(a) Oper. 1 w.r.t. $\{u, v\}$ **(b)** Oper. 2 w.r.t. f **(c)** Oper. 3 w.r.t. $G_{2,3,4}$ **(d)** Oper. 4 w.r.t. G_2

Figure 3 The four operations that are the building blocks for our piecewise-linear morphs.

For two drawings Γ_1 and Γ_2 of a graph, we denote by $\langle \Gamma_1, \Gamma_2 \rangle$ the linear morph between Γ_1 and Γ_2. We now state a result about 2D morphs.

▶ **Theorem 2** (Alamdari et al. [3]). *Let G be an n-vertex plane graph. There exists an $O(n)$-step crossing-free piecewise-linear 2D morph between any two planar straight-line drawings of G.*

3 An Upper Bound

This section is devoted to a proof of the following theorem.

▶ **Theorem 3.** *For any two planar straight-line drawings (not necessarily with the same embedding) of an n-vertex planar graph, there exists a crossing-free piecewise-linear 3D morph between them with $O(n^2)$ steps.*

We first assume that the given planar graph G is biconnected and describe four operations (Section 3.1) that allow us to morph a given 2D planar straight-line drawing of G into another one, while achieving some desired change in the embedding. We then show (Section 3.2) how these operations can be used to construct a crossing-free piecewise-linear 3D morph between any two planar straight-line drawings of G. Finally, we remove our biconnectivity assumption (Section 3.3).

3.1 3D Morph Operations

We begin by describing four operations that morph a given planar straight-line drawing into another with a different embedding; see Figure 3.

▶ **Lemma 4** (Operation 1: Graph flip). *Let G be a biconnected plane graph, let u and v be two distinct vertices of G, and let Γ be a planar straight-line drawing of G. Then there exists a 2-step crossing-free piecewise-linear 3D morph from Γ to a planar straight-line drawing Γ'' of G whose embedding is the flip of the embedding that G has in Γ; moreover, u and v do not move during the morph.*

Proof. We implement Operation 1, which proves the lemma, as follows. Refer to Figure 4. Let Π be the plane $z = 0$, which contains Γ. Let Π' be the plane that is orthogonal to Π

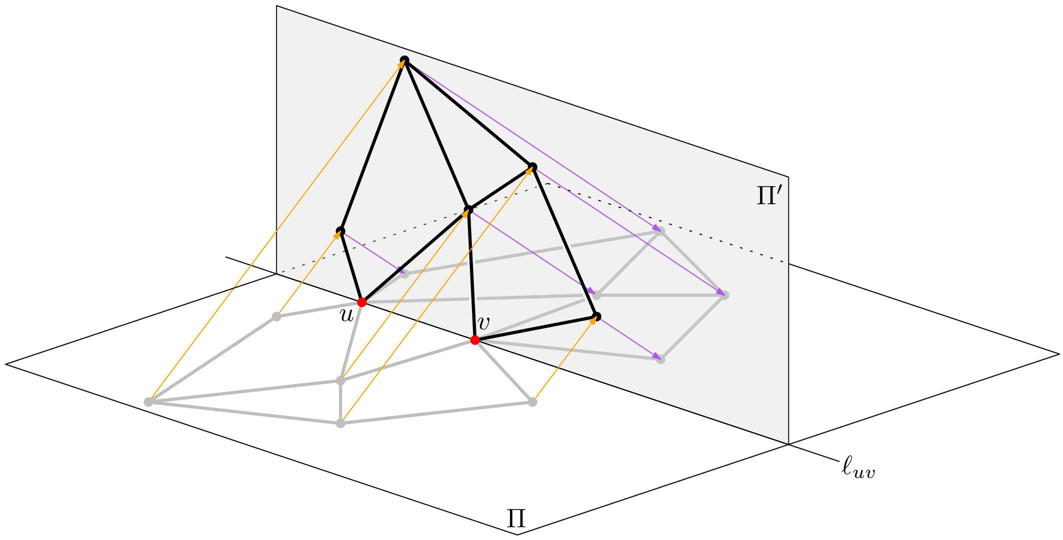


Figure 4 Illustration for Operation 1. The drawings Γ and Γ'' are gray, while Γ' is black. Vertex trajectories in the linear morphs $\langle\Gamma, \Gamma'\rangle$ and $\langle\Gamma', \Gamma''\rangle$ are represented by arrowed lines.

and contains the line ℓ_{uv} through u and v. Let Γ' be the image of Γ under a clockwise rotation around ℓ_{uv} by 90°. Note that Γ' is contained in Π'. Now let Γ'' be the image of Γ' under another clockwise rotation around ℓ_{uv} by 90°. Note that Γ'' is a flipped copy of Γ and is contained in Π. Consider the linear morphs $\langle\Gamma, \Gamma'\rangle$ and $\langle\Gamma', \Gamma''\rangle$. In each of them, every vertex travels on a line that makes a 45°-angle with both Π and Π', and all these lines are parallel. Due to this, due to the linearity of the morphs, and due to the fact that, in both morphs, pre-image and image are planar, all vertices stay coplanar during both linear morphs. (As in a true rotation, the intermediate drawing lies on a plane that contains ℓ_{uv} and rotates around it; unlike in a true rotation, the size of the intermediate drawing changes continuously.) In particular, every intermediate drawing is crossing-free, and u and v (as well as all the points on ℓ_{uv}) are fixed points. ◀

▶ **Lemma 5** (Operation 2: Outer face change)**.** *Let G be a biconnected plane graph, let Γ be a planar straight-line drawing of G, and let f be a face of Γ. Then there exists a 4-step crossing-free piecewise-linear 3D morph from Γ to a planar straight-line drawing Γ''' of G whose embedding is the same as the one of Γ, except that the outer face of Γ''' is f.*

Proof. We implement Operation 2, which proves the lemma, using the stereographic projection. Let Π be the plane $z = 0$, which contains Γ. Let S be a sphere that contains Γ in its interior and is centered on a point in the interior of f; see Figure 5. Let Γ' be the 3D straight-line drawing obtained by projecting the vertices of G from their positions in Γ vertically to the Northern hemisphere of S. Let Γ'' be determined by projecting the vertices of Γ' centrally from the North Pole of S to Π. Both projections define linear morphs: $\langle\Gamma, \Gamma'\rangle$ and $\langle\Gamma', \Gamma''\rangle$. Indeed, any intermediate drawing is crossing-free since the rays along which we project are parallel in $\langle\Gamma, \Gamma'\rangle$ and diverge in $\langle\Gamma', \Gamma''\rangle$, and there is a one-to-one correspondence between the points in the pre-image and in the image. Since the morph also inverts the rotation system of Γ'' with respect to Γ, we apply Operation 1 to Γ'', which, within two morphing steps, flips Γ'' and yields our final drawing Γ'''. ◀

Consider a split pair $\{u, v\}$ of G and a drawing Γ of G in which u and v are incident to the outer face, as in Figure 6a. Let $G_1, \ldots, G_k$ be the split components of G with respect

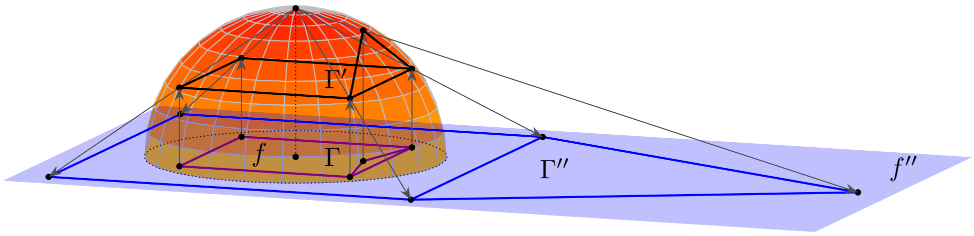
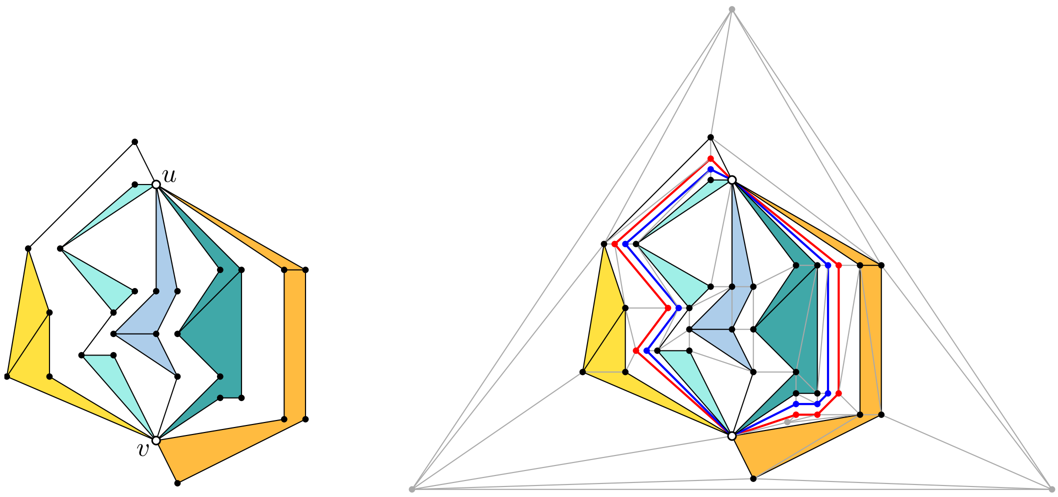


Figure 5 The first two steps of Operation 2 are projections. First, the vertices of the drawing Γ (purple) are projected vertically to a hemisphere that is centered on some interior point of a given inner face f. This yields a new drawing Γ' (black). From there, the vertices are projected centrally from the North Pole back to the plane $z = 0$ resulting in a drawing Γ'' (blue), where f'', the image of f, is the outer face.



(a) Drawing Γ of G.

(b) Drawing Ψ of H; polygon P_{in} is blue, P_{out} is red.

Figure 6 Illustration for Operation 3 with $i = 2$ and $j = 4$: Construction of Ψ from Γ.

to $\{u, v\}$ in clockwise order around u such that G_1 and G_k are incident to the outer face of Γ. We say that a pair (i, j) with $1 \le i \le j \le k$ is *good* if it has the following property: If G contains the edge (u, v), then this edge is one of the components $G_i, \dots, G_j$.[1]

Operation 3 allows us to flip the embedding of the components $G_i, \dots, G_j$ (and to incidentally reverse their order), while leaving the embedding of the other components of G unchanged. This is formalized in the following.

▶ **Lemma 6** (Operation 3: Component flip). *Let G be a biconnected plane graph, and let $\{u, v\}$ be a split pair of G. Let Γ be a planar straight-line drawing of G in which u and v are incident to the outer face. Let $G_1, \dots, G_k$ be the split components of G with respect to $\{u, v\}$ in clockwise order around u such that G_1 and G_k are incident to the outer face of Γ. Let (i, j) be a good pair.*

Then there exists an $O(n)$-step crossing-free piecewise-linear 3D morph from Γ to a planar straight-line drawing Γ' of G in which, for $\ell \in \{i, \dots, j\}$, the embedding of G_ℓ is the flip of its

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[1] This is a point where our operations differ from the ones of Angelini et al. [4]. Indeed, their flip operation applies to any sequence of components of G, while ours does not.

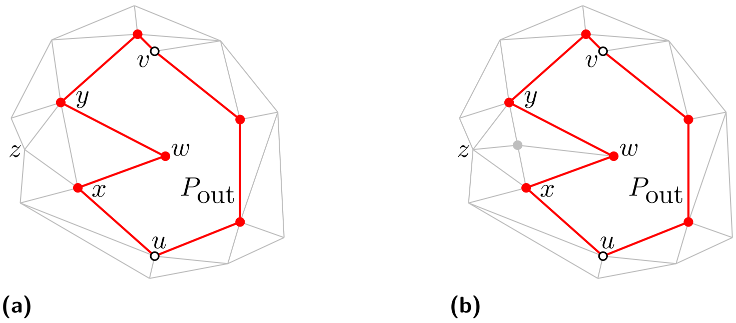


Figure 7 Triangulating the exterior of P_{out} (only P_{out} and the triangles incident to vertices of P_{out} are shown). This can create chords with respect to P_{out} (see (x, y) in (a)), which are then treated as in (b).

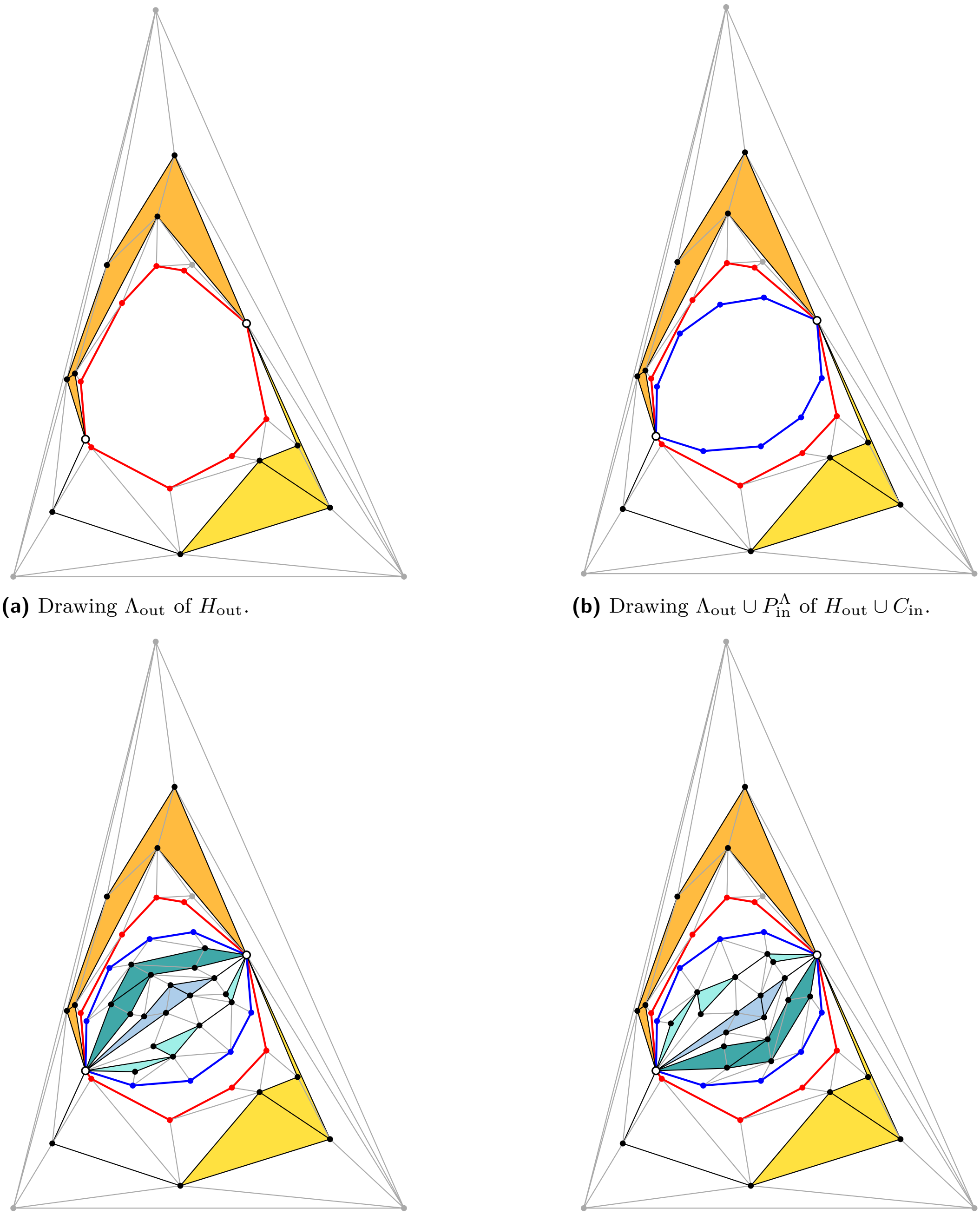
embedding in Γ*, while the embedding of* G_ℓ *is the same as in* Γ*, for* $\ell \in \{1, \dots, i-1, j+1, \dots, k\}$*. The order of* $G_1, \dots, G_k$ *around* u *in* Γ' *is* $G_1, \dots, G_{i-1}, G_j, G_{j-1}, \dots, G_i, G_{j+1}, \dots, G_k$*.*

Proof. In order to implement Operation 3, which proves the lemma, ideally we would like to apply Operation 1 to the drawing of the graph $G_i \cup G_{i+1} \cup \dots \cup G_j$ in Γ. However, this would result in a drawing that may contain crossings between edges of $G_i \cup G_{i+1} \cup \dots \cup G_j$ and edges of the rest of the graph. Thus, we first move $G_i \cup G_{i+1} \cup \dots \cup G_j$, via a crossing-free piecewise-linear 2D morph, into a polygon that is symmetric with respect to the line through u and v and that does not contain any edges of the rest of the graph. Applying Operation 1 to $G_i \cup G_{i+1} \cup \dots \cup G_j$ now results in a drawing in which $G_i \cup G_{i+1} \cup \dots \cup G_j$ still lies inside the same symmetric polygon, which ensures that the edges of $G_i \cup G_{i+1} \cup \dots \cup G_j$ do not cross the edges of the rest of the graph.

We now describe the details of Operation 3; refer to Figure 6b. We start by drawing a triangle (a, b, c) surrounding Γ. Then we insert in Γ two polygons P_{in} and P_{out} with $O(n)$ vertices, which intersect Γ only at u and v; the vertices of $G_1, \dots, G_{i-1}, G_{j+1}, \dots, G_k$ (except u and v) and a, b, and c lie outside P_{out}; the vertices of $G_i, \dots, G_j$ (except u and v) lie inside P_{in}; P_{out} contains P_{in}; and the two paths of P_{in} connecting u and v have the same number of vertices. We let P_{in} and P_{out} "mimic" the boundary of the drawing of $G_i \cup G_{i+1} \cup \dots \cup G_j$ in Γ.

We triangulate the exterior of P_{out}; that is, we triangulate each region inside (a, b, c) and outside P_{out} bounding a face of the current drawing. If this introduces a chord (x, y) with respect to P_{out}, as in Figure 7a, let (x, y, w) and (x, y, z) be the two faces incident to (x, y); we subdivide (x, y) with a vertex and connect this vertex to w and z, as in Figure 7b. We also triangulate the interior of P_{in}. Let Ψ be the resulting planar straight-line drawing of this plane graph H. Let C_{out} and C_{in} be the cycles of H represented by P_{out} and P_{in} in Ψ, let H_{out} be the subgraph of H induced by the vertices that lie outside or on P_{out}, and let H_{in} be the subgraph of H induced by the vertices that lie inside or on P_{in}. Note that H_{out} is a triconnected plane graph, as each of its faces is delimited by a 3-cycle, except for one face, which is delimited by a cycle C_{out} without chords. Further, H_{in} is an internally-triconnected plane graph, as each of its internal faces is delimited by a 3-cycle, while the outer face is delimited by a cycle C_{in} which may have chords.

We now construct another planar straight-line drawing of H, as follows. Construct a strictly convex drawing Λ_{out} of H_{out} by means of Theorem 1, as in Figure 8a. Let $P^{\Lambda}_{\mathrm{out}}$ be the strictly convex polygon representing C_{out} in Λ_{out}. As in Figure 8b, plug a strictly convex drawing $P^{\Lambda}_{\mathrm{in}}$ of C_{in} in the interior of $P^{\Lambda}_{\mathrm{out}}$ (except at u and v) so that $P^{\Lambda}_{\mathrm{in}}$ is symmetric with



(a) Drawing Λ_{out} of H_{out}.

(b) Drawing $\Lambda_{\text{out}} \cup P^{\Lambda}_{\text{in}}$ of $H_{\text{out}} \cup C_{\text{in}}$.

(c) Drawing Λ of H.

(d) Drawing Φ of H.

Figure 8 Illustration for Operation 3.

respect to the line through u and v. This can be achieved because the two paths of C_{in} connecting u and v have the same number of vertices and because P^{Λ}_{out} is strictly convex, hence the segment $\overline{uv}$ lies in its interior, and thus also a polygon P^{Λ}_{in} sufficiently close to $\overline{uv}$ does. Finally, plug into $\Lambda_{\text{out}} \cup P^{\Lambda}_{\text{in}}$ a strictly convex drawing Λ_{in} of H_{in} in which C_{in} is represented by P^{Λ}_{in}, as in Figure 8c; this drawing can be constructed again by Theorem 1. This results in a planar straight-line drawing Λ of H.

We now describe the morph that occurs in Operation 3. We first define a piecewise-linear morph $\langle \Psi, \ldots, \Phi \rangle$ from Ψ to another planar straight-line drawing Φ of H, as the concatenation of two piecewise-linear morphs $\langle \Psi, \ldots, \Lambda \rangle$ and $\langle \Lambda, \ldots, \Phi \rangle$. The morph $\langle \Psi, \ldots, \Lambda \rangle$ is an $O(n)$-step crossing-free piecewise-linear 2D morph obtained by Theorem 2. The morph $\langle \Lambda, \ldots, \Phi \rangle$ is an $O(1)$-step piecewise-linear 3D morph that is obtained by applying Operation 1 to Λ_{in} only, with u and v fixed; Figure 8d shows the resulting drawing Φ. In order to prove that Operation 3 defines a crossing-free morph, it suffices to observe that, during $\langle \Lambda, \ldots, \Phi \rangle$, the intersection of H_{in} with the plane on which Λ_{out} lies is (a subset of) the segment $\overline{uv}$, which lies in the interior of a face of Λ_{out}; hence, H_{in} does not intersect H_{out} (except in u and v). That no other crossings occur during $\langle \Psi, \ldots, \Phi \rangle$ is a consequence of Theorem 2 (which ensures that $\langle \Psi, \ldots, \Lambda \rangle$ has no crossings) and of the properties of Operation 1 (which ensure that $\langle \Lambda, \ldots, \Phi \rangle$ has no crossings among the edges of H_{out}). Finally, Operation 3 is the piecewise-linear morph $\langle \Gamma, \ldots, \Gamma' \rangle$ obtained by restricting the morph $\langle \Psi, \ldots, \Phi \rangle$ to the vertices and edges of G. Note that the effect of Operation 1, applied only to Λ_{in}, is the one of flipping the embeddings of $G_i, \ldots, G_j$ (and also reversing their order around u), while leaving the embeddings of $G_1, \ldots, G_{i-1}, G_{j+1}, \ldots, G_k$ unaltered, as claimed. ◀

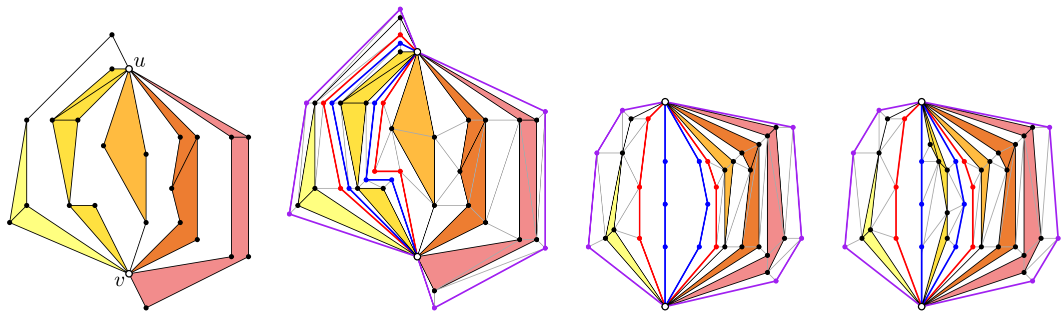
Operation 4 works in a setting similar to the one of Operation 3; see Figure 9a. Operation 4 allows one component to "skip" the other components of G, so to be incident to the outer face. This is formalized in the following.

▶ **Lemma 7** (Operation 4: Component skip). *Let G, $G_1, \ldots, G_k$, $\{u, v\}$, and Γ be defined as in Operation 3. If the edge (u, v) exists, then G_1 must be (u, v). Let $i \in \{2, \ldots, k\}$.*

Then there exists an $O(n)$-step crossing-free piecewise-linear 3D morph from Γ to a planar straight-line drawing Γ' in which, for $\ell \in \{1, \ldots, k\}$, the embedding of G_ℓ is the same as in Γ, and the clockwise order of the split components around u is $G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_k, G_i$, where G_1 and G_i are incident to the outer face.

Proof. In order to implement Operation 4, which proves the lemma, we would like to first move G_i vertically up from the plane $z = 0$ to the plane $z = 1$, to then send G_i "far away" by modifying the x- and y-coordinates of its vertices, and to finally project G_i vertically back to the plane $z = 0$. There are two complications to this plan, though. The first one is given by the vertices u and v, which belong both to G_i and to the rest of the graph. When moving u and v on the plane $z = 1$, the edges incident to them are dragged along, which may result in these edges crossings each other. The second one is that there may be no far away position that allows the drawing of G_i to be vertically projected back to the plane $z = 0$ without introducing any crossings. This is because the rest of the graph may be arbitrarily mingled with G_i in the initial drawing Γ. As in Operation 3, convexity comes to the rescue. Indeed, we first employ a crossing-free piecewise-linear 2D morph which makes the boundary of the outer face of G convex and moves G_i into a convex polygon. After moving G_i vertically up to the plane $z = 1$, sending G_i far away can be simply implemented as a scaling operation, which ensures that the edges incident to u and v do not cross each other during the motion of G_i on the plane $z = 1$ and that projecting G_i vertically back to the plane $z = 0$ does not introduce crossings with the edges of the rest of the graph.

We now provide the details of Operation 4, which works slightly differently depending on whether the edge (u, v) exists and not. We first describe the latter case; see Figure 9b. We insert two polygons P_{in} and P_{out} with $O(n)$ vertices in Γ. As in Operation 3, they intersect Γ only at u and v, with P_{in} inside P_{out} (except at u and v). All the vertices of G_i (except u and v) lie inside P_{in} and all the vertices of $G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_k$ (except u and v) lie



(a) Drawing Γ of G.

(b) Drawing Ψ of H.

(c) Drawing $\Lambda_{\text{out}} \cup P^{\Lambda}_{\text{in}}$ of $H_{\text{out}} \cup C_{\text{in}}$.

(d) Drawing Λ of H.

Figure 9 Illustration for Operation 4: P_{in} is blue, P_{out} is red, and P_{ext} is purple.

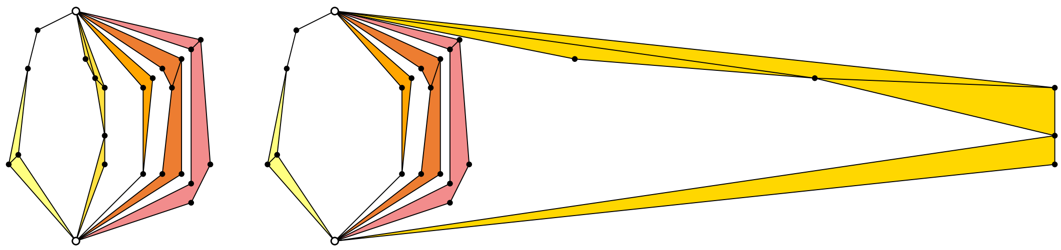
outside P_{out}. We also insert in Γ a polygon P_{ext}, with $O(n)$ vertices, that intersects Γ only at u and v, and that contains all the vertices of G and P_{out} (except u and v) in its interior.

We now triangulate the region inside P_{ext} and outside P_{out}, without introducing chords for P_{out}. We also triangulate the interior of P_{in} without introducing chords for P_{in}. Let Ψ be the resulting planar straight-line drawing of a plane graph H. Let C_{out}, C_{in}, and C_{ext} be the cycles of H represented by P_{out}, P_{in}, and P_{ext} in Ψ, respectively. Let H_{out} be the subgraph of H induced by the vertices that lie outside or on P_{out}. Similarly, let H_{in} be the subgraph of H induced by the vertices that lie inside or on P_{in}. Note that H_{out} is an internally-triconnected plane graph and H_{in} is a triconnected plane graph.

We now construct another planar straight-line drawing of H, as follows. First, construct a strictly convex drawing Q_{ext} of C_{ext} such that the angle of Q_{ext} at u (and the angle at v) is cut by the segment $\overline{uv}$ into two angles both smaller than 90°. Next, construct a strictly convex drawing Λ_{out} of H_{out} in which C_{ext} is represented by Q_{ext}, by means of Theorem 1. Let P^{Λ}_{out} be the strictly convex polygon representing C_{out} in Λ_{out}. As in Figure 9c, plug a strictly convex drawing P^{Λ}_{in} of C_{in} in the interior of P^{Λ}_{out}, except at u and v, so that the path π_{uv} that is traversed when walking in clockwise direction along C_{in} from u to v is represented by the straight-line segment $\overline{uv}$. Finally, plug into $\Lambda_{\text{out}} \cup P^{\Lambda}_{\text{in}}$ a convex drawing Λ_{in} of H_{in} in which the outer face is delimited by P^{Λ}_{in}, by means of Theorem 1. This results in a planar straight-line drawing Λ of H; see Figure 9d.

We now describe the morph that occurs in Operation 4. We first define a piecewise-linear morph $\langle \Psi, \dots, \Phi \rangle$ from Ψ to an "almost" planar straight-line drawing Φ of H, as the concatenation of two piecewise-linear morphs $\langle \Psi, \dots, \Lambda \rangle$ and $\langle \Lambda, \dots, \Phi \rangle$. The morph $\langle \Psi, \dots, \Lambda \rangle$ is an $O(n)$-step crossing-free piecewise-linear 2D morph obtained by Theorem 2. Translate and rotate the Cartesian axes so that, in Λ, the y-axis passes through u and v and u has a smaller y-coordinate than v. The morph $\langle \Lambda, \dots, \Phi \rangle$ is a 3-step piecewise-linear 3D morph defined as follows.

- The first morphing step $\langle \Lambda, \Lambda' \rangle$ moves all the vertices of H_{in}, except for u and v, vertically up, to the plane $z = 1$. As the projection to the plane $z = 0$ of every intermediate drawing of H in $\langle \Lambda, \Lambda' \rangle$ coincides with Λ, the morph is crossing-free.
- The second morphing step $\langle \Lambda', \Lambda'' \rangle$ is such that Λ'' coincides with Λ', except for the x-coordinates of the vertices of H_{in}, which are all multiplied by the same real value $s > 0$. We choose the value s large enough so that, in Λ'', the following statements hold true: (i) The absolute value of the slope of the line through u and through the projection to the plane $z = 0$ of any vertex of H_{in} not in π_{uv} is smaller than the absolute value of the slope



(a) Drawing obtained by restricting Λ to G. **(b)** Drawing Γ' obtained by restricting Φ to G.

Figure 10 Illustration for Operation 4: Construction of Γ' from the restriction of Λ to G.

of every edge incident to u in H_{out}; and (ii) a symmetric statement holds with respect to v instead of u. This morph is crossing-free since it just scales the drawing of H_{in}, while leaving the drawing of H_{out} unaltered. Intuitively, this is the step where G_i "skips" $G_{i+1}, \dots, G_k$ (although $G_i - \{u, v\}$ still lies on a different plane than $G_{i+1}, \dots, G_k$).

- The third morphing step $\langle \Lambda'', \Phi \rangle$ moves the vertices of H_{in} vertically down, to the plane $z = 0$. This morphing step may actually have crossings in its final drawing Φ. However, the property on the slopes guaranteed by the second morphing step ensures that the only crossings are those involving edges incident to vertices of π_{uv} different from u and v, which do not belong to G. Hence, the restriction of $\langle \Lambda'', \Phi \rangle$ to G is a crossing-free morph.

As in Operation 3, the actual crossing-free piecewise-linear morph $\langle \Gamma, \dots, \Gamma' \rangle$ is obtained by restricting the morph $\langle \Psi, \dots, \Phi \rangle$ to G, see Figure 10.

We now discuss the case that the edge (u, v) exists; then G_1 is such an edge. Now P_{in} and P_{out} surround all the components $G_1, \dots, G_i$, and not just G_i; consequently, H_{in} comprises $G_1, \dots, G_i$. The description of Operation 4 remains the same, except for two differences. First, P^{Λ}_{in} is strictly convex; in particular, the path π_{uv} is not represented by a straight-line segment, so that the edge (u, v) lies in the interior of P^{Λ}_{in}. Second, in the 3-step piecewise-linear 3D morph $\langle \Lambda, \Lambda', \Lambda'', \Phi \rangle$, not all the vertices of H_{in} are lifted to the plane $z = 1$, then scaled, and then projected back to the plane $z = 0$, but only those of G_i. The arguments for the fact that the restriction of such a morph to G is crossing-free remain the same. ◀

3.2 3D Morphs for Biconnected Planar Graphs

We now describe an algorithm that constructs an $O(n^2)$-step piecewise-linear morph between any two planar straight-line drawings Γ and Φ of the same n-vertex biconnected planar graph G. It actually suffices to construct an $O(n^2)$-step piecewise-linear morph from Γ to *any* planar straight-line drawing Λ of G with *the same embedding* as Φ, as then an $O(n)$-step piecewise-linear morph from Λ to Φ can be constructed by means of Theorem 2. And even more, it suffices to construct an $O(n^2)$-step piecewise-linear morph from Γ to *any* planar straight-line drawing Ψ of G that has *the same rotation system* as Λ, as then an $O(1)$-step piecewise-linear morph from Ψ to Λ can be constructed by means of Operation 2.

As proved by Di Battista and Tamassia [8], starting from a planar graph drawing (in our case, Γ), one can obtain the rotation system of any other planar drawing (in our case, Φ) of the same graph by: (i) suitably changing the permutation of the components in some *parallel compositions*; that is, for some split pairs $\{u, v\}$ that define three or more split components, changing the clockwise (circular) ordering of such components; and (ii) flipping the embedding

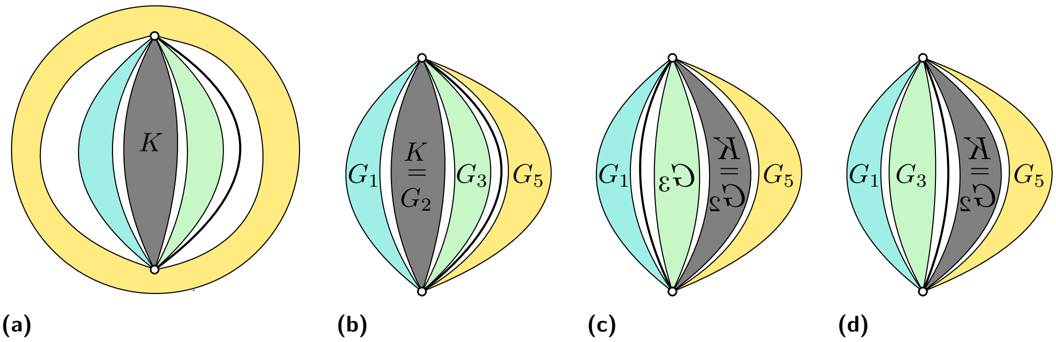


Figure 11 How the flip the embedding of a split component K in a drawing Γ, which is sketched in (a). We first apply Operation 2 to morph Γ so that its outer face is incident to u and v, as in (b). In this illustration, we have that G_2 and G_4 respectively coincide with K and with the edge (u, v), hence $\ell = 2$ and $m = 4$. Applying Operation 3 with $i = \ell = 2$ and $j = m = 4$ flips the embedding of G_2, G_3, and G_4, as in (c). Applying Operation 3 again with $i = \ell = 2$ and $j = m - 1 = 3$ leaves only the embedding of K flipped with respect to the original embedding, as in (d).

for some *rigid compositions*; that is, for some split pairs that define a maximal split component that is biconnected, flipping the embedding of the component. Thus, it suffices to show how to implement these modifications by means of Operations 1–4 from Section 3.1. We first take care of the flips, not only in the description, but also algorithmically: All the flips are performed before all the permutation rearrangements since the flips may cause some changes regarding the permutations, which we fix later.

Let $\{u, v\}$ be a split pair that defines a maximal biconnected split component K of G, and suppose that we want to flip the embedding of K in Γ. (The drawing we deal with undergoes modifications, however for the sake of simplicity we always denote it by Γ.) The drawing Γ is shown in Figure 11a. Note that K is not the edge (u, v), as otherwise we would not need to flip its embedding. Further, K does not properly contain (u, v) since (u, v) would be a split component by itself. However, (u, v) may belong to $E(G) - E(K)$. Apply Operation 2 to morph Γ so that the outer face becomes any face incident to u and v, as in Figure 11b. Let $G_1, \dots, G_k$ be the split components of G with respect to $\{u, v\}$, in clockwise order around u, where G_1 and G_k are incident to the outer face. Let $\ell \in \{1, \dots, k\}$ be such that $G_\ell = K$. We distinguish two cases, depending on whether the edge (u, v) belongs to G or not.

- If the edge (u, v) does not belong to G, then we simply apply Operation 3, with $i = j = \ell$, in order to morph Γ to flip the embedding of $G_\ell = K$.
- If (u, v) belongs to G, then let $m \in \{1, \dots, k\}$ be such that G_m is (u, v). Assume that $\ell < m$, the other case is symmetric. Apply Operation 3 with $i = \ell$ and $j = m$, in order to morph Γ such that the embeddings of $G_\ell, G_{\ell+1}, \dots, G_m$ will be flipped as in Figure 11c. If we again let $G_1, \dots, G_k$ denote the split components of G with respect to $\{u, v\}$, in clockwise order around u, where G_1 and G_k are incident to the outer face, G_ℓ is now the edge (u, v) and G_m is K. Apply Operation 3 a second time, with $i = \ell$ and $j = m - 1$, in order to morph Γ such that the embeddings of $G_\ell, G_{\ell+1}, \dots, G_{m-1}$ will be flipped back to the embeddings they originally had; see Figure 11d. As desired, only the embedding of $K = G_m$ is actually flipped.

Flipping the embedding of K is hence done in $O(n)$ morphing steps. Since there are $O(n)$ maximal biconnected split components whose embedding may need to be flipped, all such flips are performed in $O(n^2)$ morphing steps.

Let $\{u, v\}$ be a split pair of G that defines three or more split components, and suppose that we want to change the clockwise (circular) ordering of such components around u to a different one. If the edge (u, v) exists, then apply Operation 2 to morph Γ so that the outer face becomes the one to the left of (u, v), when traversing (u, v) from u to v; otherwise, apply Operation 2 to morph Γ so that the outer face becomes any face incident to u and v. Let $G_1, \dots, G_k$ be the split components of G with respect to $\{u, v\}$, in clockwise order around u, where G_1 and G_k are incident to the outer face; note that, if (u, v) exists, then it coincides with G_1. Let $G_1, G_{\sigma(2)}, G_{\sigma(3)}, \dots, G_{\sigma(k)}$ be the desired clockwise order of the split components of G with respect to $\{u, v\}$ around u; since we are only required to fix a clockwise *circular* order of these components, we can assume G_1 to be the first component in the desired clockwise *linear* order of such components around u that starts at the outer face.

We apply Operation 4 with index $\sigma(2)$, then again with index $\sigma(3)$, and so on until the index $\sigma(k)$. The first j applications make $G_{\sigma(2)}, G_{\sigma(3)}, \dots, G_{\sigma(j+1)}$ the last j split components of G with respect to $\{u, v\}$ in the clockwise linear order of the components around u that starts at the outer face. Hence, after the last application we obtain the desired order. Each application of Operation 4 requires $O(n)$ morphing steps, hence changing the clockwise order around u of the k split components of G with respect to a split pair $\{u, v\}$ takes $O(nk)$ morphing steps, where k is the number of split components with respect to $\{u, v\}$. Since the total number of split components with respect to every split pair of G that defines a parallel composition is in $O(n)$ [8], this sums up to $O(n^2)$ morphing steps. This concludes the proof of Theorem 3 for biconnected planar graphs.

3.3 3D Morphs for General Planar Graphs

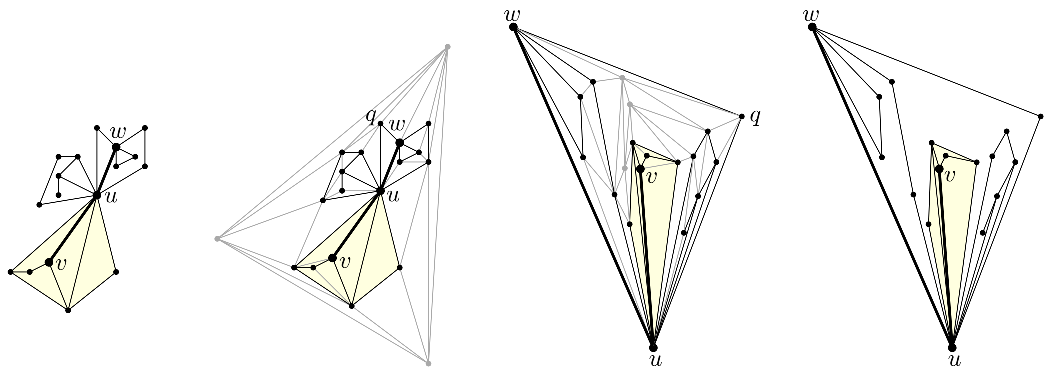
We start by reducing the general problem to the one in which G is connected. Suppose that G has multiple connected components $G_1, \dots, G_k$. Let Γ_1 and Γ_2 be the two given planar straight-line drawings of G between which we want to construct a crossing-free piecewise-linear 3D morph. For $i = 1, \dots, k$ and for $j = 1, 2$, let $\Phi_{i,j}$ be the restriction of Γ_j to G_i.

Assume that, for $i = 1, \dots, k$, we know how to construct a crossing-free piecewise-linear 3D morph $\mathcal{M}_i = \langle \Phi_{i,1}, \dots, \Phi_{i,2} \rangle$; how to construct $\mathcal{M}_i$ will be explained later in the section. Assume that the entire morph $\mathcal{M}_i$ happens within a ball with diameter $\mathcal{W}_i$. Clearly, this is true for a sufficiently large value $\mathcal{W}_i > 0$. Let $\mathcal{W} = \max_{i=1,\dots,k} \mathcal{W}_i$.

Note that we cannot just construct a morph between Γ_1 and Γ_2 as the union of the morphs $\mathcal{M}_i$, as distinct connected components of G would collide with one another during such a morph. Hence, our strategy is to move each connected component to a distinct horizontal plane. These planes are sufficiently far apart from each other so that morphs of individual components do not interfere with one another.

More in detail, we proceed as follows. For $j = 1, 2$, let $\langle \Gamma_j, \Psi_j \rangle$ be the crossing-free piecewise-linear 3D morph that moves the drawing $\Phi_{i,j}$ of G_i in Γ_j vertically up to the plane $\mathcal{P}_i$ with equation $z = 3i\mathcal{W}$. Now distinct connected components of G lie on different horizontal planes, both in Ψ_1 and in Ψ_2. Moreover, each connected component G_i lies on the same plane, namely $\mathcal{P}_i$, in Ψ_1 as in Ψ_2.

We can now obtain a morph $\langle \Psi_1, \dots, \Psi_2 \rangle$ as the union of the morphs $\mathcal{M}_i$ (the morph $\mathcal{M}_i$, rather than between the drawings $\Phi_{i,1}$ and $\Phi_{i,2}$, actually occurs between the drawings $\Psi_{i,1}$ and $\Psi_{i,2}$ which are the vertical translation of $\Phi_{i,1}$ and $\Phi_{i,2}$ to the plane $\mathcal{P}_i$). Since the distance between any two planes $\mathcal{P}_i$ is larger than or equal to $3\mathcal{W}$ and since each morph $\langle \Psi_{i,1}, \dots, \Psi_{i,2} \rangle$ happens within a ball with diameter $\mathcal{W}$, distinct connected components of G do not collide with each other during $\langle \Psi_1, \dots, \Psi_2 \rangle$. Thus, $\langle \Gamma_1, \Psi_1, \dots, \Psi_2, \Gamma_2 \rangle$ is the desired



(a) Drawing Γ; block B is yellow. **(b)** Triangulating Γ into a drawing Ψ. **(c)** Changing the outer face of Ψ. **(d)** Drawing Γ'.

Figure 12 Illustration for the morph that allows the path (v, p, w) to be inserted in Γ.

crossing-free piecewise-linear 3D morph between Γ_1 and Γ_2.

Notice that the number of steps of such a morph is two plus the number of steps needed to morph a single connected component of G, as distinct connected components of G are morphed simultaneously during the morph $\langle \Psi_1, \dots, \Psi_2 \rangle$.

We now assume that G is connected; let Γ and Φ be the prescribed planar straight-line drawings of G we want to morph. We are going to augment Γ and Φ to planar straight-line drawings of a biconnected planar graph and then apply the algorithm of Section 3.2. The augmentation is done in $k-1$ steps, where k is the number of biconnected components of G. At each step, the augmentation decreases by one the number of biconnected components of G by employing $O(n)$ morphing steps. Thus, the total number of morphing steps used by the augmentation is in $O(n^2)$. We now describe how a single augmentation step is done (the drawing Γ and the graph G we deal with undergo some modifications, however for the sake of simplicity we always denote them by Γ and G).

Let B be a biconnected component of G that contains a unique cut-vertex u (that is, B is a leaf of the block-cut-vertex tree of G [9, 11]). Let (u, v) and (u, w) be two edges that are consecutive in the clockwise order of the edges incident to u in Φ and such that $(u, v) \in E(B)$ and $(u, w) \notin E(B)$. We are going to augment G with a length-2 path (v, p, w), thus decreasing the number of biconnected components of G. Such a path can be planarly inserted in Φ, because of the way v and w were defined. However, v and w are not necessarily incident to the same face of Γ, as in Figure 12a; in order to allow for a planar insertion of the path (v, p, w), we are going to let v and w share a face by suitably morphing Γ.

Triangulate Γ into a planar straight-line drawing Ψ of a maximal planar graph H, as in Figure 12b, and then apply Operation 2 to morph Ψ in $O(1)$ steps to change its outer face into any of the two faces incident to the edge (u, w), as in Figure 12c; let q be the third vertex incident to such a face. By means of Theorem 1, we construct a planar straight-line drawing Λ of H in which the cycle (u, w, q) delimiting the outer face is represented by a triangle whose angle at u is smaller than 45°. An $O(n)$-step crossing-free piecewise-linear 2D morph from Ψ to Λ can be obtained by Theorem 2. Restricting such morphs to G provides an $O(n)$-step crossing-free piecewise-linear morph from Γ to a planar straight-line drawing Γ' of G contained inside a triangle (u, w, q) whose angle at u is smaller than 45°, as in Figure 12d.

Translate and rotate the Cartesian axes so that the origin is at u and the positive y-halfaxis cuts the interior of the face that is to the right of the edge (u, v), when traversing such an edge from u to v. We are now ready to make u and v incident to the same face in Γ'. This is done in three morphing steps.

- The first morphing step moves all the vertices of B, except for u, vertically up, to the plane $z = 1$.
- The second morphing step scales the x and y-coordinates of all the vertices of B by a vector (α, β), where α and β are two positive real values satisfying the following properties: (i) β is sufficiently large so that every vertex in $V(B) - \{u\}$ has a y-coordinate larger than the one of every vertex in $V(G) - V(B)$; and (ii) α is large enough so that the slope of every edge (u, r) of B is either between 0° and 45° (if r has positive x-coordinates) or between 135° and 180° (if r has negative x-coordinates).
- The third morphing step moves all the vertices of B vertically down, back to the plane $z = 0$.

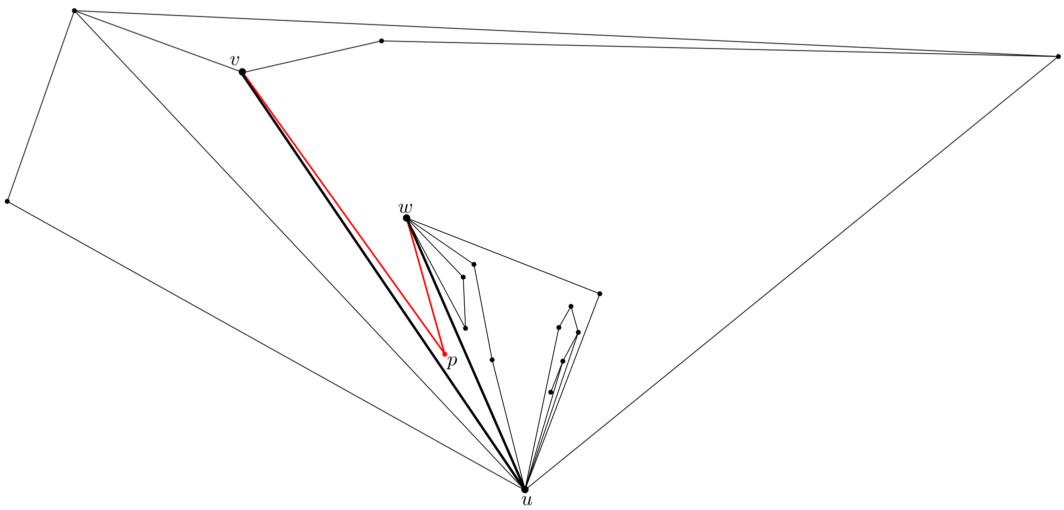


Figure 13 Illustration for the morph that allows the path (v, p, w) to be inserted in Γ. Scaling B up so that it surrounds the rest of the graph.

The first two morphing steps are clearly crossing-free. The third one is also crossing-free, because of the properties that are ensured by the choice of α and β in the second morphing step. Now v and w are incident to the same face not only in Φ, but also in Γ'. Thus, they can be connected via a length-2 path (v, p, w); the new vertex p can be inserted close to u, both in Γ' and in Φ, as in Figure 13. Now B and the biconnected component w used to belong to have been merged into a single biconnected component, as desired.

4 Discussion: Lower Bounds

Though the algorithm of Section 3 uses a quadratic number of steps, we are not aware of any super-constant lower bound for crossing-free piecewise-linear 3D morphs between planar straight-line graph drawings. The nested-triangles graph provides a linear lower bound on the number of steps required for a crossing-free piecewise-linear *2D* morph, as proved by Alamdari et al. [3]. Specifically, let T_k be the pair of drawings of the graph that consists of $k + 1$ nested triangles, connected by three paths that are spiraling in the first drawing and straight in the second drawing, as in Figure 14 for $k = 3$. The lower bound of Alamdari et al. [3] relies on the fact that the innermost triangle or the outermost triangle makes a linear number of full turns in any crossing-free piecewise-linear 2D morph between the two drawings.

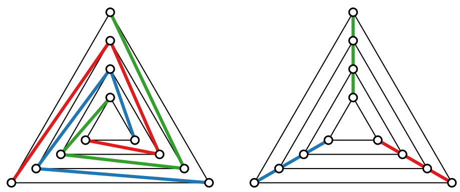


Figure 14 The lower bound example of Alamdari et al. [3].

Even in 3D, it may seem that a linear number of linear morphs is required. However, the extra dimension allows us to perform the "turns" in parallel by "flipping" several triangles at once. The key operation is to morph T_6 in a constant number of steps without moving the innermost and outermost triangles, as shown in Figure 15 and animated in [1, 2]. Then for any k, we can construct a crossing-free piecewise-linear 3D morph between the two drawings in T_{6k} in a constant number of steps by performing the morph of Figure 15 in parallel for the k nested copies of T_6. Observe that in this morph the $(6i+1)$-th outermost triangle does not move, for any $i = 0, \dots, k$. Each morphing step of T_6 avoids a small tetrahedron above and below its innermost triangle, allowing different nested copies of T_6 to morph in parallel without intersecting.

The above example gives hope that the number of steps required to construct a crossing-free piecewise-linear 3D morph between any two given planar straight-line graph drawings could be far smaller than quadratic – potentially even constant. However, it is unclear how to generalize our procedure.

The approach of Figure 15 relies on the sequence of nested triangles to be *independent*, as we can untangle each one locally without affecting the others. This is not necessarily the case. For instance, the example in Figure 16 shows a tree of nested triangles that are recursively twisted by 120° at every level. Here, each path in the tree has the same structure as a nested-triangles graph thus, in total, it requires $\Omega(\log n)$ morphing steps in 2D. It is unclear to us how to handle the dependencies between different tree branches.

5 Open Problems

Our research raises several other open problems. An immediate one is to reduce our quadratic upper bound for the number of steps that are needed to construct a crossing-free piecewise-linear 3D morph between any two planar straight-line graph drawings. Extending the result of Arseneva et al. [5], we ask whether planar graph families richer than trees, e.g., outerplanar graphs and series-parallel graphs, admit crossing-free piecewise-linear 3D morphs with a sub-linear number of steps.

We have given an example of two topologically equivalent planar straight-line drawings of a triconnected graph that can be untangled in 3D using only $O(1)$ steps. Still we think that there are examples of planar graphs with topologically equivalent drawings where this is not the case. More specifically, we suspect that in 3D, as in 2D, a linear number of steps is sometimes necessary.

If the initial configuration can also make use of the third dimension, the initial configuration can be an arbitrary knot, and the final configuration can be a regular polygon in the plane. Then, a morph exists only if the initial configuration is the unknot, but this condition may

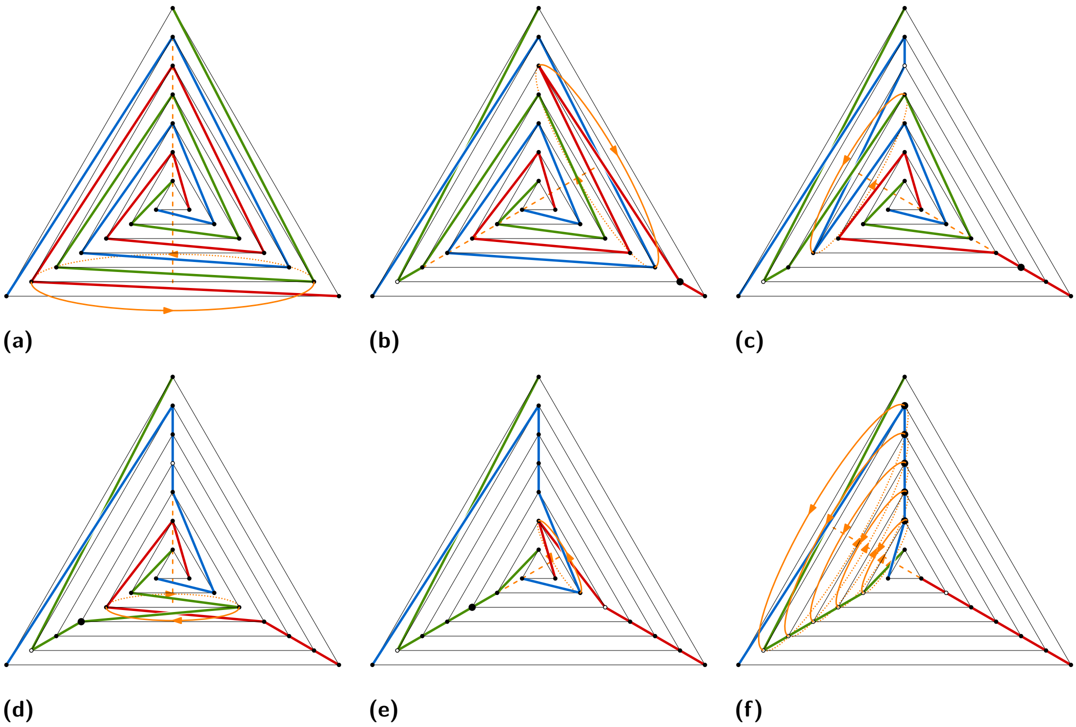


Figure 15 Morphing T_6 in 3D without moving the innermost and outermost triangles. Orange arrows show the vertices that exchange position in the next step. Empty / large disks indicate that a vertex lies below / above the plane containing the initial drawing. The drawing obtained by the morph is of the type of the right drawing in Figure 14.

not be sufficient because our edges must remain straight during the morph. That is, it may be necessary to insert extra vertices (e.g. by subdividing edges) before a (topological) unknot can actually be unknotted by one of our morphs. It is unclear whether extra vertices are necessary, and whether polynomially many extra vertices are sufficient.

Acknowledgements. The research for this paper started at the Dagstuhl Seminar 22062: "Computation and Reconfiguration in Low-Dimensional Topological Spaces". The authors thank the organizers and the other participants for a stimulating atmosphere and interesting discussions.

References

1 Nested spiral example: constant number of linear morphs. `https://www.geogebra.org/m/djmqqhst`.
2 Nested triangles/spiral example: constant number of linear morphs. `https://vimeo.com/718624499`.
3 Soroush Alamdari, Patrizio Angelini, Fidel Barrera-Cruz, Timothy M. Chan, Giordano Da Lozzo, Giuseppe Di Battista, Fabrizio Frati, Penny Haxell, Anna Lubiw, Maurizio Patrignani, Vincenzo Roselli, Sahil Singla, and Bryan T. Wilkinson. How to morph planar graph drawings. *SIAM J. Comput.*, 46(2):824–852, 2017. `doi:10.1137/16M1069171`.
4 Patrizio Angelini, Pier Francesco Cortese, Giuseppe Di Battista, and Maurizio Patrignani. Topological morphing of planar graphs. *Theor. Comput. Sci.*, 514:2–20, 2013. `doi:10.1016/j.tcs.2013.08.018`.

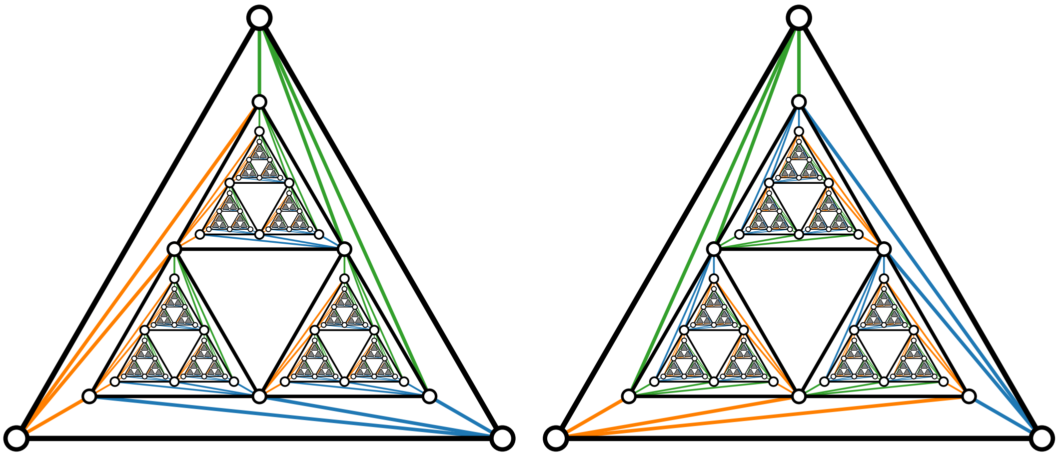


Figure 16 A potential lower bound construction.

5 Elena Arseneva, Prosenjit Bose, Pilar Cano, Anthony D'Angelo, Vida Dujmović, Fabrizio Frati, Stefan Langerman, and Alessandra Tappini. Pole dancing: 3D morphs for tree drawings. *J. Graph Algorithms Appl.*, 23(3):579–602, 2019. doi:10.7155/jgaa.00503.

6 Elena Arseneva, Rahul Gangopadhyay, and Aleksandra Istomina. Morphing tree drawings in a small 3D grid. *J. Graph Algorithms Appl.*, 27(4):241–279, 2023. doi:10.7155/jgaa.00623.

7 Stewart S. Cairns. Deformations of plane rectilinear complexes. *Amer. Math. Monthly*, 51(5):247–252, 1944. doi:10.1080/00029890.1944.11999082.

8 Giuseppe Di Battista and Roberto Tamassia. On-line planarity testing. *SIAM J. Comput.*, 25(5):956–997, 1996. doi:10.1137/S0097539794280736.

9 Frank Harary. *Graph Theory.* Addison-Wesley Pub. Co. Reading, MA, 1969.

10 Seok-Hee Hong and Hiroshi Nagamochi. Convex drawings of hierarchical planar graphs and clustered planar graphs. *J. Discrete Algorithms*, 8(3):282–295, 2010. doi:10.1016/j.jda.2009.05.003.

11 John E. Hopcroft and Robert E. Tarjan. Algorithm 447: Efficient algorithms for graph manipulation. *Comm. ACM*, 16(6):372–378, 1973. doi:10.1145/362248.362272.

12 Carsten Thomassen. Deformations of plane graphs. *J. Combin. Theo. Ser. B*, 34(3):244–257, 1983. doi:10.1016/0095-8956(83)90038-2.

13 William T. Tutte. How to draw a graph. *Proc. London Math. Soc.*, 3(13):743–767, 1963. doi:10.1112/plms/s3-13.1.743.