

Quadrangulating a Polygon with a Fixed Spirality using Steiner Points

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Abstract

Let P be an n -sided polygon on the plane, where $n \geq 4$. A *quadrangulation* of P is a geometric plane graph obtained from P by adding edges to the interior of P so that every finite face is quadrilateral. It is easy to see that P does not always admit a quadrangulation. So Ramaswami et al. [15] introduced “Steiner points”, which are auxiliary points added to P which helps quadrangulatability of P . Those points are said to be *inner* and *outer* if they are added to the interior and the exterior of P , respectively. They proved that every P with Steiner points added admits a quadrangulation, and estimated the number of those Steiner points by n , when all Steiner points are inner, and when all Steiner points are outer. On the other hand, Hidaka et al. [8] used the notion of “spirality of P ” which measures how far P is from being convex, and considered how a polygon P of spirality k admits a quadrangulation with *inner* Steiner points. In this paper, we consider the quadrangulatability of a polygon of P of spirality k when all Steiner points are outer, and when both inner and outer Steiner points are allowed.

Keywords and phrases polygon, quadrangulation, Steiner point, spiral

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1 Introduction

We consider only a *geometric* plane graph G , i.e., a plane graph such that all vertices of G are distinct points on the plane, and that all edges are straight segments. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G respectively, where we denote the *order* of G (i.e., the number of vertices) by $|G|$.

Let $n \geq 3$ be an integer, and an *n -sided polygon* P is a planar geometric drawing of a cycle of length n . A *triangulation* of P is a geometric plane graph with vertex set $V(P)$ such that its outer cycle coincides with P and that each finite face is triangular.

The following fact is well-known:

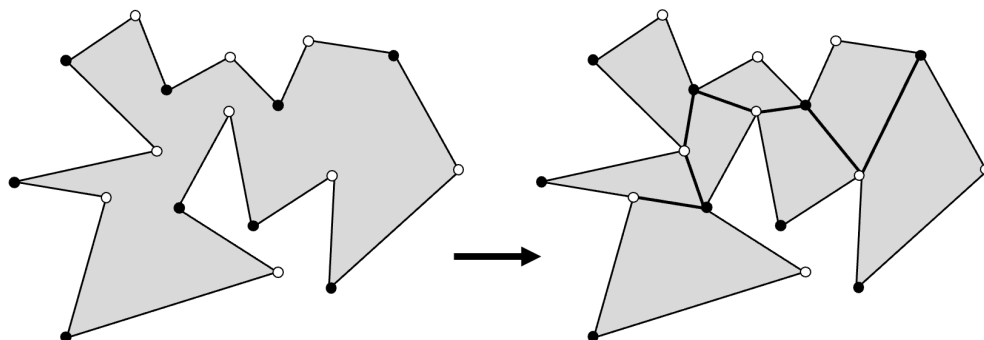
► **Proposition 1.** *Every n -sided polygon with $n \geq 3$ admits a triangulation.*

Proposition 1 is an important lemma for the well-known Fisk’s proof of *Art Gallery Theorem* [6], which states that every n -sided art gallery can be guarded by at most $\lfloor \frac{n}{3} \rfloor$ watchmen, and the estimation is known to be the best possible. See [14] for the detail.

In this paper, we focus on a *quadrangulation* of an n -sided polygon P . By an easy parity argument, we see that if P admits a quadrangulation, then n must be an even integer at least 4.

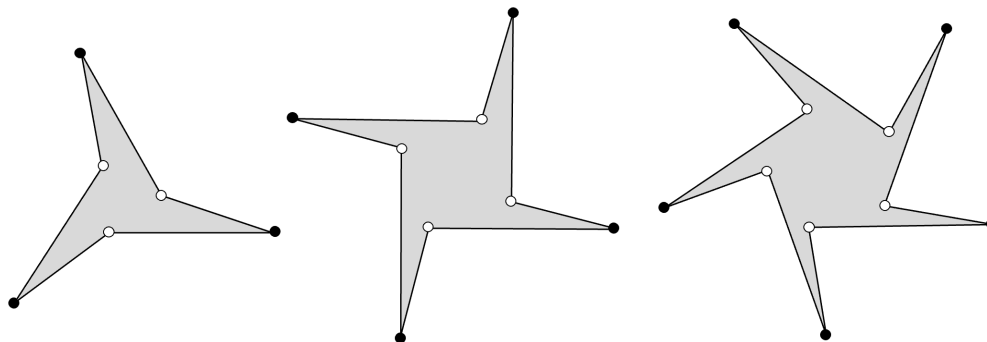


We call an n -sided polygon with even (resp., odd) n simply an *even-sided* (resp., *odd-sided*) polygon. Let us consider whether a given even-sided polygon P admits a quadrangulation. Figure 1 shows an example of an even-sided polygon P and a quadrangulation of P . We say that P is *quadrangulatable* if P admits a quadrangulation. It is known that if P consists of only horizontal and vertical segments, then P is quadrangulatable [11].



■ **Figure 1** Even-sided polygon P and a quadrangulation of P , in which we color the vertices of P black and white alternately along the boundary. Since every quadrangulation G of P is bipartite, every edge of G joins black and white vertices of P . Constructing a quadrangulation of P , this black-white coloring of P is often helpful.

However, we immediately encounter a number of counterexamples, that is, even-sided polygons which are not quadrangulatable, as shown in Figure 2. (If a polygon P admits a quadrangulation, then we have to put an edge in the interior of P joining black and white vertices, but we can add no such diagonal to these polygons.)



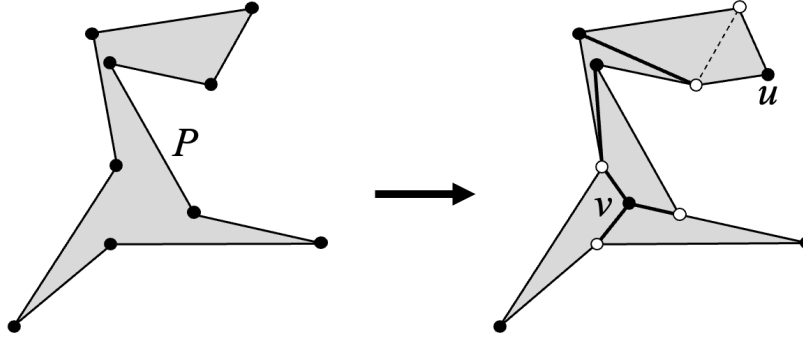
■ **Figure 2** Even-sided polygons which admit no quadrangulation

Ramaswami et al. [15] introduced “Steiner points” for a polygon P as auxiliary points to help a construction of quadrangulations of P . That is, a *Steiner point* is a point put in any position of the interior or the exterior of P , where the former is an *inner* Steiner point and the latter *outer*. Let S be a set of Steiner points for P , where we put $S = S_I \cup S_O$, and S_I and S_O are inner and outer Steiner points. We say that a polygon P is *quadrangulatable with a set S of Steiner points* if there exists a plane geometric graph G with vertex set $V(P) \cup S$ such that

- (i) the outer cycle of G is obtained from P by the following procedures: for each $v \in S_O$, we replace an edge $xy \in E(P)$ with the path xvy , and
- (ii) every finite face of G is quadrilateral.

We call G a *quadrangulation* of P with Steiner points S .

Figure 3 shows that an odd-sided polygon P and its quadrangulation with one inner Steiner point v and one outer Steiner point u . Note that we do not add an outer Steiner point to an edge whose endpoint is a Steiner point. As far as we use an outer Steiner point, we do not need a restriction of a polygon P to be even-sided, and an odd-sided polygon requires at least one outer Steiner point for its quadrangulation.



■ **Figure 3** A quadrangulation of a polygon P with an inner Steiner point v and an outer Steiner point u

Ramaswami et al. [15] proved the the following two results, in which one is for quadrangulations of a polygon with inner Steiner points, and the other is one with outer Steiner points, respectively:

► **Theorem 2** (Ramaswami et al. [15]). *Every n -sided polygon with $n \geq 4$ even admits a quadrangulation with at most $\lfloor \frac{n-2}{4} \rfloor$ inner Steiner points.*

► **Theorem 3** (Ramaswami et al. [15]). *Every n -sided polygon with $n \geq 3$ admits a quadrangulation with at most $\lfloor \frac{n}{3} \rfloor$ outer Steiner points. Moreover, the bound for the number of those points is best possible.*

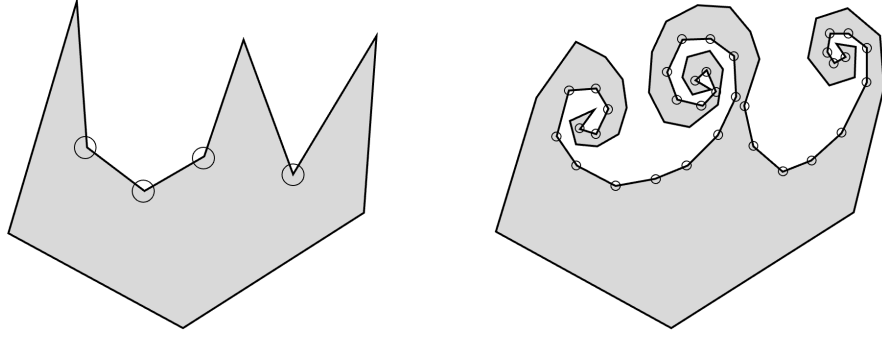
On the other hand, Nakamoto et al. [13] defined the “spirality” of a polygon P which describes a shape of P , as follows.

Let $P = v_1 \cdots v_n$ be an n -sided polygon, where $V(P) = \{v_1, \dots, v_n\}$. We say that v_i is *convex* (resp., *reflex*) if the inner angle at v_i is less (resp., greater) than π , where we suppose that no v_i has inner angle equal to π . The interval $[v_p, \dots, v_q]$ with $p \leq q$ is a *spiral* if v_i is reflex for $i = p, p+1, \dots, q$ but v_{p-1} and v_{q+1} are convex. The *spirality* of P is the number of spirals of P (see Figure 4). We note that if P has spirality k , then $|P| \geq 2k$, since there is at least one convex vertex between two consecutive spirals in P . (A similar notion measuring convexity is known [1], but this seems to be different from ours.) A motivation for defining the spirality is to point out that the number of Steiner points for quadrangulating a polygon is essentially controlled by this invariant.

Hidaka et al. [8] proved the following theorem which estimates the number of inner Steiner points of a polygon of a fixed spirality.

► **Theorem 4** (Hidaka et al. [8]). *Let P be an even-sided polygon of spirality $k \geq 1$. Then P admits a quadrangulation with at most $2k - 2$ inner Steiner points. Moreover, the estimation for the number of those points is best possible for every k .*

Figure 5 shows that the estimation in Theorem 4 is best possible, since the polygon P with spirality k requires at least $2k - 2$ inner Steiner points for its quadrangulation. We note

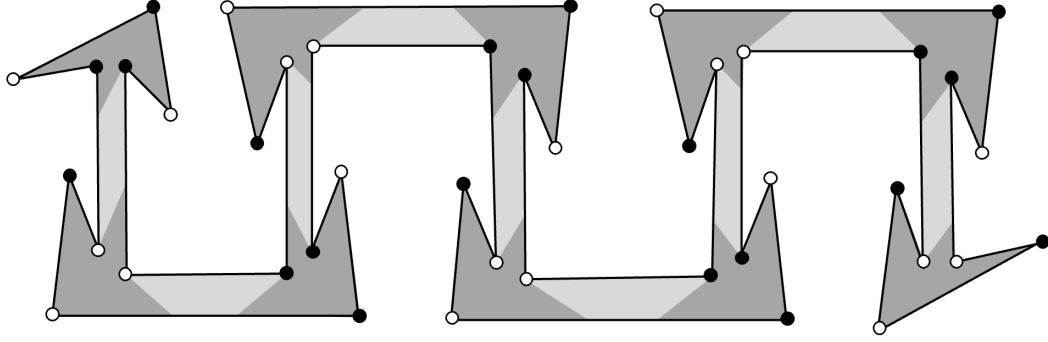


■ **Figure 4** Two polygons of spirality 2, where the marked vertices are reflex

that the examples in Figure 5 shows a tightness of Theorem 2 as well, as in the following. Since the polygon P with spirality k has $n = 8k - 6$ vertices and requires at least $2k - 2$ inner Steiner points, the number of inner Steiner points S for P is

$$|S| \geq 2k - 2 = 2 \cdot \frac{n + 6}{8} - 2 = \frac{n - 2}{4},$$

though Ramaswami et al. did not give such an example in [15].



■ **Figure 5** A polygon of spirality $k = 6$ requiring at least inner $2k - 2$ Steiner points for its quadrangulation. Each of the shaded parts requires at least one inner Steiner point, and this polygon has $2k - 2$ pairwise disjoint such parts.

Theorem 4 asserts that an even-sided polygon P is quadrangulatable (with no Steiner points) if P has spirality at most one, which were firstly proved in [13]. Theorem 4 seems to be more interesting than Theorem 2, since the spirality describes a shape of a polygon and since a polygon with a fixed spirality can have arbitrarily large number of vertices, as shown in Figure 4.

In this paper, we prove the following two theorems for quadrangulatability of a polygon of a fixed spirality when all Steiner points are outer, and when both inner and outer Steiner points are allowed, respectively.

► **Theorem 5.** *Let P be a polygon of spirality $k \geq 1$. Then P admits a quadrangulation with at most $2k - 1$ outer Steiner points. Moreover, the estimation for the number of those points is best possible for every k .*

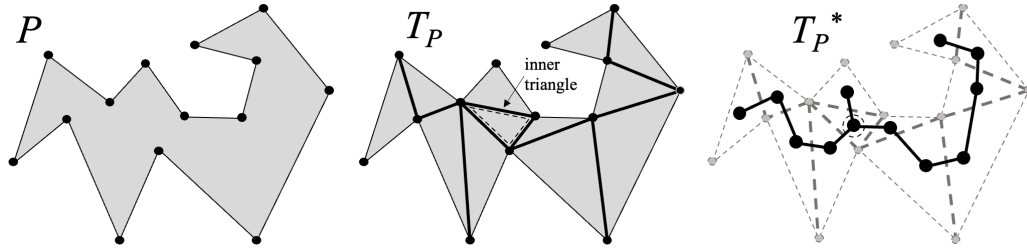
► **Theorem 6.** *Let P be a polygon of spirality $k \geq 1$. Then P admits a quadrangulation with at most k Steiner points, where inner and outer Steiner points are used. Moreover, the estimation for the number of those points is best possible for every k .*

As related topics, a triangulation of a planar point set is well-studied in the literature, and we can find many results on their generation and flip operations [4, 5, 12]. Every planar point set K admits a *triangulation*, that is, a geometric plane graph whose vertex set is K such that the outer cycle coincides with the convex hull of K , and that each finite face is triangular. However, not every point set admits a quadrangulation, and hence they sometimes need Steiner points. Those results can be found in [2, 3, 7, 10].

2 Proof of the theorems

In this section, we prove Theorems 5 and 6. Before proceeding to proving them, we consider a triangulation on a polygon, and then observe a relation between triangulations and quadrangulations of a polygon.

Let P be an n -sided polygon with $n \geq 3$. Then, by Proposition 1, P admits a triangulation, denoted by T_P . A face f of T_P is an *inner triangle* if no edge of f is contained in $E(P)$. In our argument, the inner triangles of T_P play an essential role for constructing a quadrangulation of P from T_P .



■ **Figure 6** A polygon P , a triangulation T_P of P , the graph T_P^*

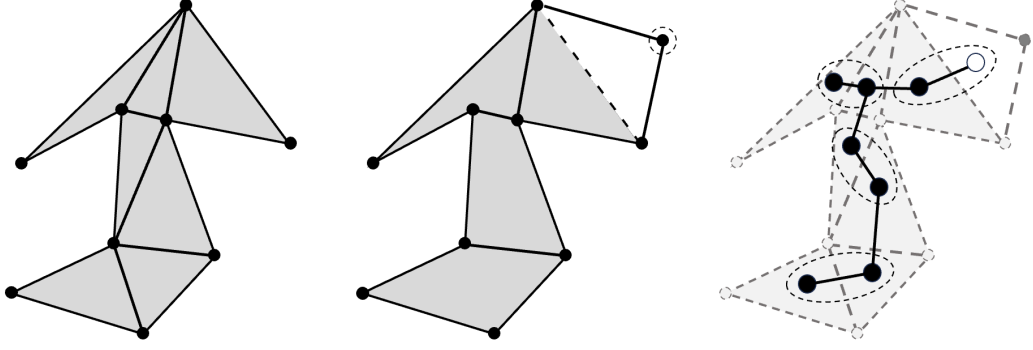
Let T_P^* denote the *dual tree* of T_P , that is, the graph whose vertices are the triangular faces of T_P , in which two vertices of T_P^* are adjacent if the corresponding two faces of T_P share an edge. Note that T_P^* is a tree of maximum degree at most 3, and that an inner triangle of T_P corresponds to a vertex of degree 3 in T_P^* .

In this section, for the simplicity of terminology, we call a tree of maximum degree at most k a k -tree, and a vertex of degree k a k -vertex. It is easy to see that a triangulation T_P of a polygon P has a spanning quadrangulation, i.e., a quadrangulation of P , if and only if the dual tree T_P^* has a perfect matching. Moreover, a triangulation T_P with m outer Steiner points added has a spanning quadrangulation if and only if T_P^* with m 1-vertices attached to a vertex of T_P^* of degree 1 or 2 has a perfect matching (see Figure 7). Note that we cannot attach a 1-vertex to a 3-vertex, since an inner triangle is not incident to an edge of P .

In order to estimate the number of outer Steiner points to be added for quadrangulating a polygon P , we need the following lemma guaranteeing the existence of a triangulation T_P of P with few inner triangles.

► **Lemma 7.** *Let P be a polygon of spirality $k \geq 1$. Then P admits a triangulation with at most $2k - 2$ inner triangles.*

The following proof of Lemma 7 is similar to that of Lemma 7 in [8], which states that a 2-colored even-sided polygon P of spirality $k \geq 1$ admits a triangulation with at most $2k - 2$ monochromatic faces. So the readers should refer it.



■ **Figure 7** Triangulation T_P , quadrangulation of P with a single outer Steiner point, and perfect matching of the dual tree

Proof of Lemma 7. Since P has a spirality k , P has k spirals $A_1, \dots, A_k$. Then removing them from P , we get k paths, say $B_1, \dots, B_k$, where each B_i consists of only convex vertices of P . Let T_P be a triangulation of P with fewest inner triangles.

Suppose that T_P has an inner triangle Δ bounded by xyz . We claim that no two vertices of Δ are contained in the same A_i nor the same B_j .

For contradictions, we firstly suppose that x and y are contained in the same spiral A_i . Then we can take the path L in A_i between x and y , where we put $L = v_1 \dots v_h$ with $x = v_1$ and $y = v_h$. Since Δ is an inner triangle, we have $h \geq 3$. By the property of A_i , the inner angle at each v_j with $j = 2, \dots, h-1$ is greater than π , and hence the straight segment xy of Δ cannot exist in the interior of P , a contradiction.

Secondly, suppose that x and y are contained in the same B_i . Similarly to the above case, B_i contains a path $L = v_1 \dots v_h$ joining $x = v_1$ and $y = v_h$, where $h \geq 3$. Since $v_1, \dots, v_h$ are all convex in P , the region R in T_P bounded by the cycle $v_1 \dots v_h z$ is a convex polygon. Then, replacing all edges in the interior of R with zv_j with $j = 2, \dots, h-1$, we can eliminate Δ . Moreover, these operations create no new inner triangle, and so we get a triangulation of P which has a fewer number of inner triangles, a contradiction.

Finally observe that T_P is combinatorially a maximal outerplane graph. Contracting each A_i and each B_j into a single vertex respectively, we get a maximal outerplane graph with $2k$ vertices, denoted by $\tilde{T}_P$. By the above observation for inner triangles, every inner triangle of T_P remains in $\tilde{T}_P$. On the other hand, by an easy computation of Euler's formula, $\tilde{T}_P$ has exactly $2k - 2$ finite faces, and hence T_P has at most $2k - 2$ inner triangles. ◀

We proceed to a proof of Theorem 5, for which we show the following two propositions.

► **Proposition 8.** *Every polygon of spirality $k \geq 1$ admits a quadrangulation with at most $2k - 1$ outer Steiner points.*

Proof. Let P be a polygon of spirality k , and let T_P be a triangulation of P with fewest inner triangles. By Lemma 7, the number of inner triangles is at most $2k - 2$. Therefore, the dual tree T_P^* of T_P is a 3-tree with at most $2k - 2$ 3-vertices.

By the above observation between a quadrangulation of P and a perfect matching of T_P^* , we prove that for a 3-tree T with m 3-vertices, a tree obtained from T by adding at most $m + 1$ 1-vertices to the vertices of T has a perfect matching, by induction on m .

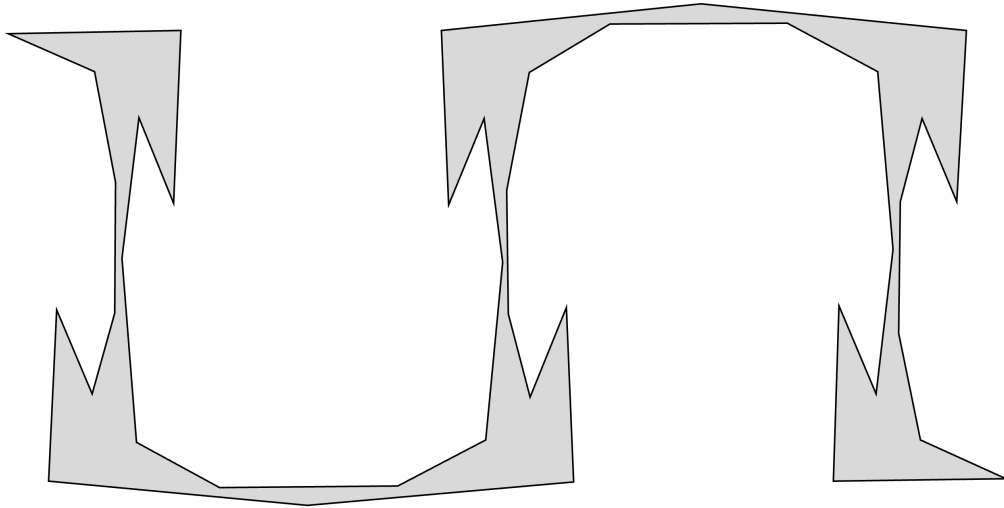
If $m = 0$, then T is a path with at least one vertex. Then, attaching at most one 1-vertex to the path, we get one with even order, which has a perfect matching.

So we may suppose $m > 0$. We can choose a 3-vertex v of T such that at least two components, say P_1 and P_2 , of $T - v$ are a path. If $|P_1|$ is even, then P_1 has a perfect matching. Otherwise, P_1 with a single 1-vertex attached has a perfect matching. The remaining graph $T' = T - V(P_1)$ is a tree with $m - 1$ 3-vertices, and hence by induction hypothesis, T' with at most m 1-vertices added has a perfect matching. Therefore, T with at most $m + 1$ 1-vertices added has a perfect matching.

Consequently, since $m \leq 2k - 2$, we can conclude that every polygon with at most $2k - 2$ inner triangles admits a quadrangulation with at most $2k - 1$ outer Steiner points. ◀

► **Proposition 9.** *For any $k \geq 1$, there exists a polygon of spirality k which requires at least $2k - 1$ outer Steiner points for its quadrangulation.*

Proof. Let P be the polygon of spirality k shown in Figure 8. The figure shows only the case when $k = 4$, but one can easily extend the shape to have any spirality $k \geq 2$. Adding m outer Steiner points to P to obtain a new polygon R , we suppose that we get a quadrangulation Q_R of R , and prove $m \geq 2k - 1$. It is easy to see that Q_R can be transformed into a triangulation, denoted by T_R , by adding a single diagonal to each face of Q_R .



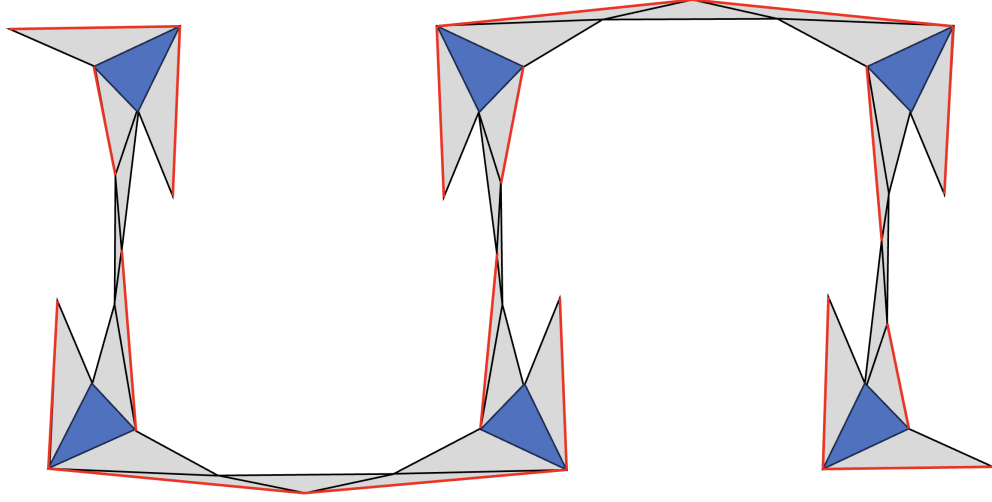
■ **Figure 8** A polygon P of spirality $k = 4$ requiring at least $2k - 1$ outer Steiner points

Let T_P be a triangulation of P , and we consider how T_R is obtained from T_P . We first give the following claim.

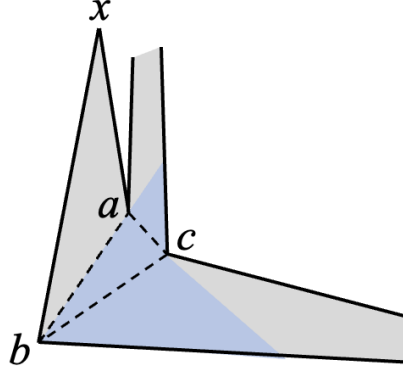
▷ **Claim 1.** The polygon P admits a unique triangulation T_P , and in particular, T_P must have $2k - 2$ inner triangles, as shown in Figure 9.

Proof. Figure 10 shows a local structure of P , where we label vertices a, b, c, x . Consider a face f of T_P containing the vertex x . Then x can see no vertex of P other than a and b , and hence f is bounded by xab in T_P . Next we consider another face f' of T_P containing the edge ab . Then a vertex of f' other than a and b must exist in the shaded region of the figure, which contains only the vertex c . Therefore, f' is the inner triangle of T_P bounded by abc . In this way, we can determine the $2k - 2$ inner triangles of T_P in P . Moreover, starting these inner triangles, we can uniquely determine all edges of T_P as shown in Figure 9. ◀

By Claim 1, we let $T_1, \dots, T_{2k-2}$ be the inner triangles in T_P , which are the shaded regions in Figure 9. Suppose that P is deformed into R by outer Steiner points, and we



■ **Figure 9** A triangulation T_P and the $2k - 2$ inner triangles



■ **Figure 10** Inner triangle T_i in T_P

consider a triangulation T_R of R . Some of $T_1, \dots, T_{2k-2}$ will still be inner triangles of T_R , and let U be the set of such inner triangles in T_P . Let

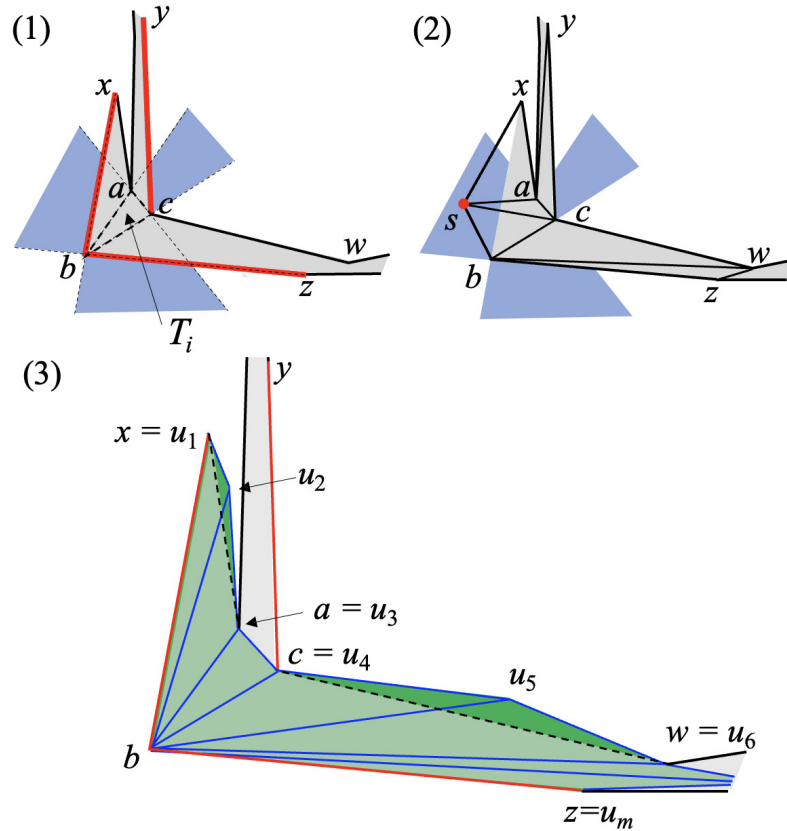
$$J = \{i \in \{1, \dots, 2k - 2\} : T_i \notin U\}.$$

Then we have the following claim.

▷ **Claim 2.** Each T_i with $i \in J$ requires at least one outer Steiner point p_i , and all the elements in $\{p_i : i \in J\}$ can be taken to be distinct.

Proof. Figure 11 (1) shows a local structure of T_i , where we label vertices a, b, c, x, y, z, w as in the figure. Putting a single outer Steiner point s in one of the shaded regions and replacing an edge of P with a path of length 2 containing s , we can break T_i in the triangulation T_R , as in Figure 11 (2). We prove that in order to break T_i in T_R , we must add an outer Steiner point to at least one of bx, bz, cy , the red edges in the figure (1).

Suppose that none of bx, bz, cy receives an outer Steiner point for R . Let us find the distinct neighbors $x = u_1, \dots, u_m = z$ of b and triangular faces $bu_i u_{i+1}$ in T_R in the clockwise order, for $i = 1, \dots, m - 1$. (See Figure 11 (3), which shows a case when a single outer Steiner point is added to each of ax and cw .) Note that each u_i is either a vertex of P or one of the



■ **Figure 11** Outer Steiner point for T_i with $i \in J$

outer Steiner points added. Observe that either $a = u_2$ or $a = u_3$ (the figure (3) shows the latter), since no outer Steiner point is added to bx and bz in R by the assumption, and since x can see only a, b and the outer Steiner point added to ax if it exists. Next, when $a = u_k$ with $k \in \{2, 3\}$, we have $u_{k+1} = c$ for the following reason. A candidate of u_{k+1} should be seen from both a and b , and hence such a point is c , or one contained in either bz or cy . So we have $c = u_{k+1}$, since the latter does not happen by the assumption. Hence we find the face T_i bounded by $bu_k u_{k+1} = bac$. Thus, if T_i is broken in T_R , then at least one of the three edges bx, bz, cy , called the *subdivide candidates* for T_i , contains an outer Steiner point, say p_i .

Finally we consider all p_i for $i \in J$. Observe that for any distinct $i, j \in J$, the subdivide candidates for T_i and those for T_j are disjoint in P (as in Figure 9). Hence all the outer Steiner points p_i with $i \in J$ are distinct. $\triangleleft$

We first suppose $J = \emptyset$, that is, all inner triangles $T_1, \dots, T_{2k-2}$ of T_P remain in T_R . Let $W = \{v_1, v_2, \dots, v_{2k-2}\}$ be the set of 3-vertices of the dual tree T_P^* corresponding to $T_1, \dots, T_{2k-2}$. Then $T_P^* - W$ has $4k - 3$ components $D_1, \dots, D_{4k-3}$, each of which has odd order. (See the left of Figure 12.) We note that adding m outer Steiner points to P to obtain R increases the number of faces of triangulations by m . Hence the order of the dual tree increases by m , but the 3-vertices $v_1, \dots, v_{2k-2}$ are fixed in this case. Hence the dual tree of T_R , denoted by T_R^* , is obtained from T_P^* by modifying each D_i with $i = 1, \dots, 4k - 3$, so that T_R^* has a perfect matching. Since each D_i has odd order, D_i requires at least one of the m vertices added or at least one vertex in W , as shown in the right of Figure 12. Hence we

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have the following estimation:

$$\begin{aligned}
 m &\geq o(P - W) - |W| \\
 &= (4k - 3) - (2k - 2) \\
 &= 2k - 1,
 \end{aligned}$$

where $o(\cdot)$ denotes the number of components of the graph with odd order.

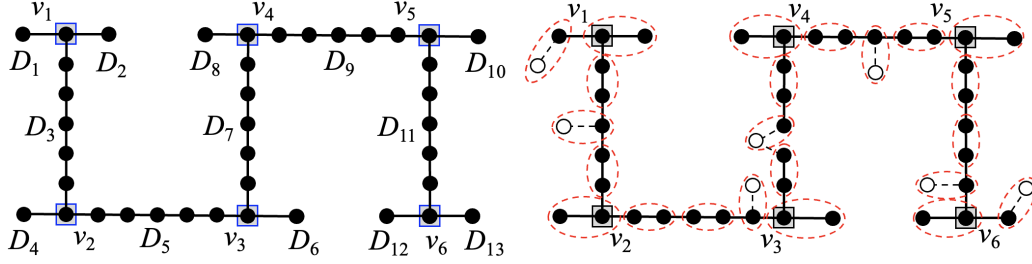


Figure 12 The dual tree T_P^* and a perfect matching of T_R^*

Secondly suppose that $J \neq \emptyset$. Let U^* be the set of 3-vertices of T_P^* corresponding to the elements of U , where we note that $U^* \cup \{v_i : i \in J\} = W$. Observe that U^* is a cut set of T_P^* and that $T_P^* - U^*$ has exactly $(4k - 3) - 2|J|$ components. Let L be any component of $T_P^* - U^*$, and suppose that L has h 3-vertices v_i with $i \in J$, where $h \geq 0$. Then L consists of such h 3-vertices and $2h + 1$ odd paths of $T_P^* - W$, and hence

$$|L| = h + (2h + 1) = 3h + 1.$$

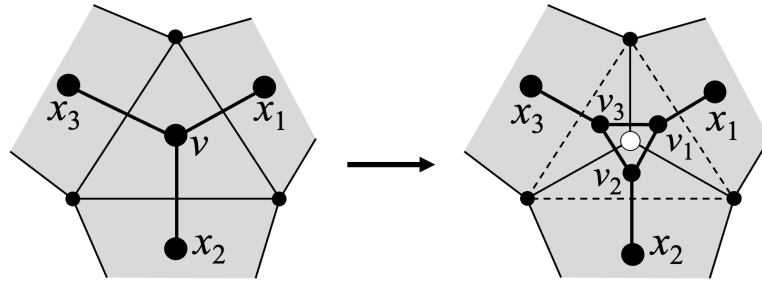
By Claim 2, L receives at least h vertices to break h inner triangles T_i with $i \in J$. Since $|L| + h = (3h + 1) + h \equiv 1 \pmod{2}$, L must receive at least $h + 1$ vertices from the m vertices added or those in U^* , in order to have even order. Remember that $T_P^* - U^*$ has $(4k - 3) - 2|J|$ components, and observe that $|U^*| = 2k - 2 - |J|$, and that $\sum_L h = |J|$. Therefore, we get the following:

$$\begin{aligned}
 m &\geq \left(\sum_L (h + 1) \right) - |U^*| \\
 &= \sum_L h + \sum_L 1 - (2k - 2 - |J|) \\
 &= |J| + (4k - 3) - 2|J| - (2k - 2 - |J|) \\
 &= 2k - 1.
 \end{aligned}$$

Therefore, P requires at least $2k - 1$ outer Steiner points for its quadrangulation. $\blacktriangleleft$

Proof of Theorem 5. Theorem 5 follows from Propositions 8 and 9. $\blacktriangleleft$

Before proving Theorem 6, we verify an operation in the dual tree T_P^* of a polygon P corresponding to the addition of a single inner Steiner point to an inner triangle of a triangulation T_P . As in Figure 13, the operation is to replace a 3-vertex v incident to three vertices x_1, x_2 and x_3 by a triangle $v_1v_2v_3$ with v_i joined to x_i , for $i = 1, 2, 3$. We call this operation a *triangular expansion* of v .



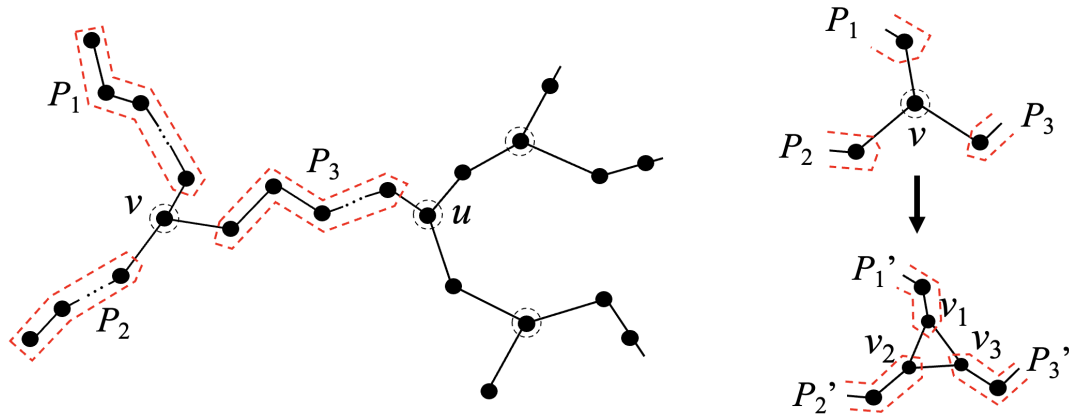
■ **Figure 13** A triangular expansion of a 3-vertex in a 3-tree

Proof of Theorem 6. The proof proceeds similarly to that of Theorem 5. For a polygon P of spirality k , we take a triangulation T_P of P with at most $2k - 2$ inner triangles, by Lemma 7. Then the dual tree T_P^* is a 3-tree with at most $2k - 2$ 3-vertices.

We prove that a 3-tree T with at most $2k - 2$ 3-vertices can be transformed into one with a perfect matching by at most k applications of either attaching a new single 1-vertex to a 1-vertex, or a triangular expansion of a 3-vertex, by induction on k .

When $k = 1$, then T has no 3-vertex, by Lemma 7, and hence T is a path. If T is a path of even order, then it has a perfect matching, while the path with a 1-vertex attached has a perfect matching if the order of T is odd. Hence we get the first step of induction.

We suppose $k > 1$, and take a 3-vertex v in T such that at least two components, say P_1 and P_2 , of $T - v$ are a path. Let u be the 3-vertex of T closest to v , and let P_3 be the path consisting of all inner vertices of the path between u and v , as shown in the left of Figure 14. We have the following four cases depending on the parity of $|P_1| + |P_2|$ and that of $|P_3|$, as follows:



■ **Figure 14** Three paths P_1, P_2 and P_3

- (1) $|P_1| + |P_2| \equiv 1 \pmod{2}$ and $|P_3| \equiv 0 \pmod{2}$
- (2) $|P_1| + |P_2| \equiv 1 \pmod{2}$ and $|P_3| \equiv 1 \pmod{2}$
- (3) $|P_1| + |P_2| \equiv 0 \pmod{2}$ and $|P_3| \equiv 0 \pmod{2}$
- (4) $|P_1| + |P_2| \equiv 0 \pmod{2}$ and $|P_3| \equiv 1 \pmod{2}$

Case (1). The path $P_1 \cup \{v\} \cup P_2$ has even order, and P_3 has even order. So both of these paths have a perfect matching.

Case (2). The path $P_1 \cup \{v\} \cup P_2$ has a perfect matching. Though P_3 has odd order, we can take a perfect matching after adding a single 1-vertex to P_3 .

Case (3). The path $P_1 \cup \{v\} \cup P_2$ has odd order, and so we add a 1-vertex to $P_1 \cup \{v\} \cup P_2$ so that the resulting path has even order. On the other hand, P_3 has even order. Therefore, these two paths have a perfect matching.

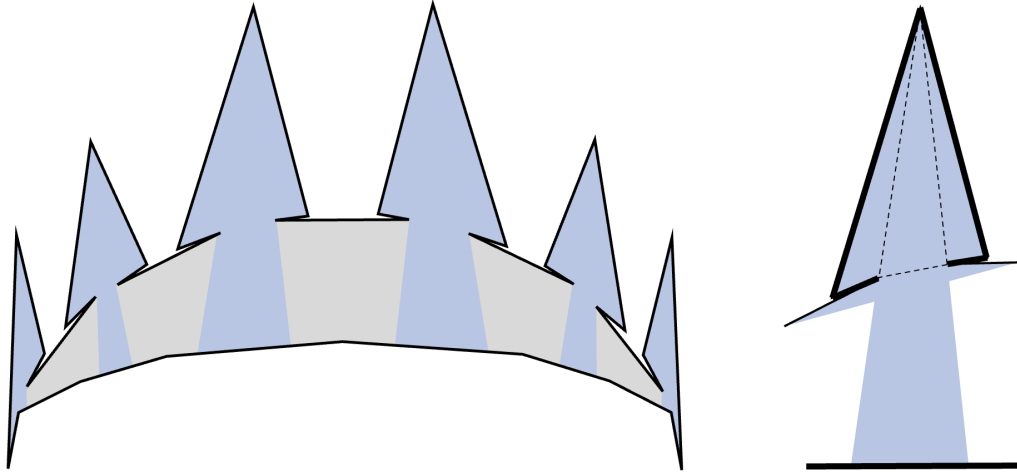
Case (4). Apply a triangular expansion of v replacing it with a triangle $v_1v_2v_3$, and join v_i to P_i to obtain the path P'_i , for $i = 1, 2, 3$, as shown in the right of Figure 14. Then $P'_1 \cup P'_2$ has even order, and so does P'_3 . Hence, these paths have a perfect matching.

In each of the above cases, adding either a single 1-vertex or applying a single triangular expansion of v , we get two paths of even order from $P_1 \cup P_2$ and P_3 respectively. Therefore, removing them, we get a 3-tree, denoted by T' , with at most $2k - 4 = 2(k - 1) - 2$ 3-vertices, since u is no longer a 3-vertex in T' . By induction hypothesis, T' can be transformed into one with a perfect matching by at most $k - 1$ applications of the two operations. Therefore, by at most k applications in total, T can be transformed into one with a perfect matching.

Consequently, every polygon of spirality k is quadrangulatable with at most k Steiner points, where we use both inner and outer Steiner points.

On the other hand, consider the polygon Q of spirality k shown in the left of Figure 15, where the figure is one with $k = 6$. Then Q has k swollen parts, and the right of Figure 15 shows a partial structure of one of the swollen parts. We can verify that each of those parts requires either an inner or outer Steiner point, as follows. It is easy to see that the swollen part is uniquely triangulated, as indicated by the dotted segments of the right figure. Moreover, in order to quadrangulate this part, we need either an inner Steiner point in the shaded region in the figure, or an outer Steiner point attached to an edge in the thick segments in the figure. Since those swollen parts are apart from each other, those shaded regions have no intersection in the polygon Q , and they share no edge for which an outer Steiner point might be added. Therefore, we have the conclusion.

Therefore, Q needs at least k Steiner points, and the estimation for the number of Steiner points in Theorem 6 is best possible. ◀



■ **Figure 15** A polygon Q of spirality k which requires at least k Steiner points

Observe that the polygon P of spirality k shown in Figure 15 has $6k$ vertices and needs at least k Steiner points for quadrangulating P . Therefore, we have the following.

► **Proposition 10.** *There exists a polygon P with n vertices which requires at least $\frac{n}{6}$ inner and outer Steiner points.*

3 Conclusion

In this paper, we consider quadrangulatability of an n -sided polygon P using Steiner points. Rawaswami et al. [15] gave a bound for the number of Steiner points needed for quadrangulating P by using n . They actually gave two theorems, Theorems 2 and 3, using inner Steiner points and outer Steiner points, respectively.

On the other hand, Nakamoto et al. defined the spirality of a polygon, which describes its shape and is independent of the number of vertices [8]. Hidaka et al. gave the best possible bound for the number of *inner* Steiner points for quadrangulating a polygon P of spirality $k \geq 1$ by using k (Theorem 4).

Our contribution in this paper is to prove the following: For quadrangulating a polygon P of spirality $k \geq 1$,

- (1) the number of *outer* Steiner points is at most $2k - 1$, and this estimation is best possible (Theorem 5).
- (2) the number of *inner* and *outer* Steiner points is at most k , and this estimation is best possible (Theorem 6).

In addition to Theorem 4, by proving (1) and (2) in this paper, we have completely solved the problems for quadrangulatability of a polygon of a fixed spirality by inner Steiner points, by outer Steiner points, and by inner and outer Steiner points.

Finally we estimate the number of inner and outer Steiner points S for quadrangulating a polygon of n vertices. By Theorem 2, we have $|S| \leq \lfloor \frac{n-2}{4} \rfloor$ for even n . For odd n , adding a single outer Steiner point, we have $S \leq \lfloor \frac{(n+1)-2}{4} \rfloor = \lfloor \frac{n-1}{4} \rfloor$ by Theorem 2. On the other hand, there is an n -sided polygon requiring $|S| \geq \frac{n}{6}$, by Proposition 10. Consequently we have Table 1.

■ **Table 1** The number of Steiner points for quadrangulating a polygon

	by the number n of vertices	by spirality k
Inner Steiner points	$\lfloor \frac{n-2}{4} \rfloor$ (Rawaswami [15])	$2k - 2$ (Hidaka [8])
Outer Steiner points	$\lfloor \frac{n}{3} \rfloor$ (Rawaswami [15])	$2k - 1$
Inner and outer	$\lfloor \frac{n}{6} \rfloor \leq \leq \lceil \frac{n-1}{4} \rceil$	k

Declarations

Data Availability. Data sharing not applicable to this paper as no datasets were generated or analyzed during the study.

Conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

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