Beyond Outerplanarity

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Abstract

We study straight-line drawings of graphs where the vertices are placed in convex position in the plane, i.e., *convex drawings*. We consider two families of graph classes with convex drawings: *outer k-planar* graphs, where each edge is crossed by at most k other edges; and *outer k-quasi-planar* graphs, where no k edges can mutually cross.

We show that the outer k-planar graphs are $\lfloor 3.5\sqrt{k} \rfloor$-degenerate, and consequently that every outer k-planar graph can be colored with $\lfloor 3.5\sqrt{k} \rfloor + 1$ colors. We further show that every outer k-planar graph has a balanced vertex separator of size at most $2k + 3$. For each fixed k, these small balanced separators allow us to test outer k-planarity in quasi-polynomial time, e.g., this implies that none of these recognition problems is NP-hard unless the Exponential Time Hypothesis fails. We also show that the class of outer 3-quasi-planar graphs and the class of planar graphs are incomparable.

Finally, we restrict outer k-planar and outer k-quasi-planar drawings to *full* drawings (where no crossing appears on the boundary of the outer face) and to *closed* drawings (where the vertex sequence on the boundary of the outer face is a Hamiltonian cycle in the graph). For each k, we express *closed outer k-planarity* and *closed outer k-quasi-planarity* in extended monadic second-order logic. Since every outer k-planar graph has treewidth $O(k)$, Courcelle's theorem implies that closed outer k-planarity is linear-time testable. We leverage this result to further show that full outer k-planarity can also be tested in linear time.

Keywords and phrases Graph Drawing, Topological Graphs, Beyond Planarity, Outer k-Planarity, Outer k-Quasi-Planarity.

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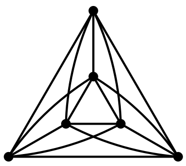
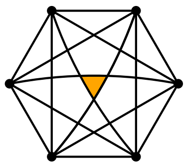
1 Introduction

A *drawing* of a graph maps each vertex to a distinct point in the plane, each edge to a Jordan curve connecting the points of its incident vertices but not containing the point of any other vertex, and two such Jordan curves have at most one common point. In the last few years, the focus in graph drawing has shifted from exploiting structural properties of planar graphs



(a) a 1-planar and, hence, quasi-planar drawing



(b) an outer 4-planar and outer 4-quasi-planar drawing

Figure 1 Two drawings of K_6. Note that K_6 is neither outer 3-planar nor outer quasi-planar.

to addressing the question of how to produce well-structured (understandable) drawings in the presence of edge crossings, i.e., to the topic of *beyond-planar* graph classes. The primary approach here has been to define and study graph classes which allow some edge crossings, but restrict the crossings in various ways, such as, for example, restricting the total number of crossings [38, 44] or even bundling the edges and restricting the number of crossings of bundles rather than of single edges [23, 24, 39]. Two of the most commonly studied such graph classes are:

1. *k-planar graphs*, that is, the graphs that can be drawn so that each edge is crossed by at most k other edges.
2. *k-quasi-planar graphs*, that is, the graphs that can be drawn so that no k pairwise non-incident edges mutually cross.

Note that the 0-planar graphs and 2-quasi-planar graphs are precisely the planar graphs. The 3-quasi-planar graphs are simply called *quasi-planar*. For example, K_6 is 1-planar and, hence, quasi-planar; see Fig. 1a. While all k-planar graphs are clearly $(k+2)$-quasi-planar, they are actually even $(k+1)$-quasi-planar [7].

In this paper we restrict the two above families of graph classes by insisting on drawings where all vertices lie on the outer face and the edges are routed in the complement of the outer face. Note that this is equivalent to placing the vertices in convex position and drawing the edges as straight-line segments. The graphs that admit such drawings without crossings are the *outerplanar* graphs. In this paper, we apply the above two generalizations of planar graphs to outerplanar graphs, which yields the classes of *outer k-planar* graphs and *outer k-quasi-planar* graphs. For example, K_6 is neither outer 3-planar nor outer quasi-planar, but outer 4-planar and, trivially, outer 4-quasi-planar (because it does not have four independent edges); see Fig. 1b. For these classes, we consider balanced separators, treewidth, degeneracy (see Section 1.2 below), coloring, edge density, and recognition.

1.1 Related work

Ringel [56] was the first to consider k-planar graphs; he showed that 1-planar graphs are 7-colorable. Twenty years later, Ringel's result was improved by Borodin [18], who showed that 1-planar graphs are in fact 6-colorable. This is tight since K_6 is 1-planar. Many additional results on 1-planarity can be found in a recent survey paper [47]. Generally, every n-vertex k-planar graph has treewidth $O(\sqrt{kn})$ [31] and at most $3.81n\sqrt{k}$ edges [2], and hence degeneracy at most $\lfloor 2 \cdot 3.81\sqrt{k} \rfloor$ and chromatic number at most $\lfloor 2 \cdot 3.81\sqrt{k} \rfloor + 1$.

Outer k-planar graphs have been considered mostly for $k \in \{0, 1, 2\}$. Of course, the outer 0-planar graphs are the classic outerplanar graphs which are well-known to be 2-degenerate and to have treewidth at most 2. It was shown that essentially every graph property can be tested efficiently on outerplanar graphs [12]. Outer 1-planar graphs are a simple subclass of planar graphs and can be recognized in linear time [11, 40]. *Full outer 2-planar graphs*, that is, outer 2-planar graphs that admit a drawing where no crossing appears on the boundary of the outer face, can be recognized in linear time [41]. General outer k-planar graphs were considered by Binucci et al. [16], who showed (among other results) that, for every k, there is a 2-tree that is not outer k-planar. Wood and Telle [61] considered a slight generalization of outer k-planar graphs in their work and showed that these graphs have treewidth $O(k)$. Outer 1-planar graphs have been compared [19] with *fan-crossing* graphs (that is, graphs that have a drawing where each edge can only cross edges with a common endpoint) and *fan-crossing-free* graphs (that is, graphs that have a drawing where no edge is crossed by two or more edges with a common endpoint). Fan-crossing and fan-crossing-free are complementary properties, in the sense that a drawing is 1-planar if and only if it is fan-crossing and fan-crossing-free. Brandenburg [19] showed that there are graphs that are simultaneously (outer-) fan-crossing and (outer-) fan-crossing-free but not (outer-) 1-planar. Angelini, Da Lozzo, Förster, and Schneck [8, 9] studied the edge density of k-planar *2-layer* layouts (where the vertices lie on two horizontal lines and every edge is a y-monotone curve). In order to admit such a layout, a graph must be bipartite, and the constraints on the placement of the vertices emphasize its bipartite structure. k-Planar layouts have also been studied in the context of *book embeddings* [5, 50], where the vertices lie (possibly in a given fixed order) on a straight line, the *spine*, and the edges are placed in a given number of halfplanes (called *pages*) whose pairwise intersection is the spine.

The k-quasi-planar graphs have been studied extensively from the perspective of edge density. Pach, Shahrokhi, and Szegedy [53] conjectured that every n-vertex k-quasi-planar graph has at most $c_k n$ edges, where c_k is a constant depending only on k. Their conjecture has been proven to hold for $k = 3$ [3] and $k = 4$ [1]. The best known general upper bound is $n(\frac{c \log n}{\log k})^{2 \log k - 4}$ [34, 35], where c is a positive constant. Edge density was also considered in the "outer" setting: Capoyleas and Pach [21] showed that every outer k-quasi-planar graph with n vertices has at most $2(k-1)n - \binom{2k-1}{2}$ edges. Nakamigawa [52] and Dress, Koolen, and Moulton [30] showed that there are outer k-quasi-planar graphs meeting this bound (if $n \geq 2k - 1$). Moreover, the outer k-quasi-planar graphs that meet this bound are exactly the maximal outer k-quasi-planar graphs. (Recall that a graph is *maximal* with respect to a given graph property if adding any edge to the graph destroys the property.) Both teams of authors actually showed that, given any two maximal outer k-quasi-planar graphs G and G' with the same vertex set but different edge sets, for every two outer k-quasi-planar drawings Γ and Γ' of G and G', respectively, whose corresponding vertices are in the same positions, there exists a sequence of local edge exchange operations (called *flips*) producing drawings $\Gamma = \Gamma_1, \Gamma_2, \ldots, \Gamma_t = \Gamma'$ such that every intermediate drawing is an outer k-quasi-planar drawing. More recently, it was shown that the *semi-bar k-visibility graphs* are outer $(k+2)$-quasi-planar [37]. Apart from these results, the outer k-quasi-planar graphs do not seem to have received much attention.

The *convex* (or *1-page book*) *crossing number* of a graph [58] is the minimum number of crossings, taken over all convex drawing. This concept has been introduced several times (see [58] for more details) and is closely related to *k-quasi-planar geometric graphs*, which are graphs with straight-line k-quasi-planar drawings [59].

The convex crossing number is NP-complete to compute [51]. However, Bannister and Eppstein [13, 14] used treewidth-based techniques (via extended monadic second order logic and Courcelle's theorem; see Theorem 15) to show that the convex crossing number can be computed in linear time. More precisely, they show that, given a graph G with n vertices and m edges, the convex crossing number of G, $\mathrm{cr}^\circ(G)$, can be computed in $O(f(\mathrm{cr}^\circ(G)) \cdot (n+m))$ time, where f is a computable function. Thus, recognizing *outer k-crossing graphs* is *fixed-parameter tractable*.

There is a natural connection between outer k-planar graphs and $(k/2)$-outerplanar graphs. Consider an outer k-planar graph G with some outer k-planar drawing and place a dummy vertex at each crossing to get a planarized graph G'. The graph G' is $(k/2)$-outerplanar. The treewidth of G' is known to be at most $3/2k$ [17]. We can construct a tree decomposition of G from the tree decomposition of G' by replacing every dummy vertex v of G' in each bag with the four endpoints of G corresponding to v. This increases the treewidth of the decomposition of G' by a factor of 4. Therefore, the treewidth of G is at most $6k$.

With more involved arguments, Wood and Telle [61] showed that every outer k-planar graph (actually every graph from a somewhat larger class of graphs) has treewidth at most $3k + 11$. Recently, this upper bound has been improved by Firman, Gutowski, Kryven, Okada, and Wolff [33] to $\lfloor 1.5k \rfloor + 2$. Strengthening the connection between outer k-planar graphs and $O(k)$-outerplanar graphs, Pyzik [55] constructed, for every k, a k-outerplanar graph of treewidth $3k$ that is also outer $2k$-planar.

1.2 Preliminaries

We briefly define the key graph theoretic concepts that we will study. Given a graph G, let $V(G)$ denote its vertex set and let $E(G)$ denote its edge set.

A graph is *d-degenerate* [49] if every subgraph of it has a vertex of degree at most d. This concept is used in greedy algorithms for coloring. Namely, a d-degenerate graph can be inductively $(d+1)$-colored by simply removing a vertex of degree at most d. A graph class is d-degenerate if every graph in the class is d-degenerate. Furthermore, a graph class which is *hereditary* (i.e., closed under taking subgraphs) is d-degenerate when every graph in that class has a vertex of degree at most d. Note that outerplanar graphs are 2-degenerate, and planar graphs are 5-degenerate.

A *separation* of a graph G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$, and no edge of G has one end in $A \setminus B$ and the other in $B \setminus A$. The set $A \cap B$ is called *separator*, and the *size* of the separation (A, B) is $|A \cap B|$. A separation (A, B) of a graph G on n vertices is *balanced* if $|A \setminus B| \leq 2n/3$ and $|B \setminus A| \leq 2n/3$. The *separation number* of a graph G is the smallest number s such that every subgraph of G has a balanced separation of size at most s. The *treewidth* of a graph was introduced by Robertson and Seymour [57]. It is a fundamental graph parameter that measures how close a graph is to being a tree. Treewidth plays a crucial role in algorithm design, as many otherwise intractable problems become efficiently solvable on graphs of bounded treewidth. Treewidth is closely related to separation number. Namely, any graph with treewidth t has separation number at most $t+1$ and, as Dvořák and Norin [32] showed, any graph with separation number s has treewidth at most $15s$. Furthermore, due to Courcelle's theorem [25] (which we cite in Theorem 15), if a graph class has bounded treewidth, it means that many problems can be solved efficiently for graphs in that class.

A *quasi-polynomial time* algorithm is one with a running time of the form $2^{\text{polylog}(n)}$, where n is the size of the input. The *Exponential Time Hypothesis (ETH)* [43] is a complexity theoretic assumption defined as follows. For $k \geq 3$, let

$$s_k = \inf\{\lambda \colon \text{there is an } O(2^{\lambda n})\text{-time algorithm to solve } k\text{-SAT}\}.$$

The ETH states that for $k \geq 3$, $s_k > 0$. Hence, for example, there is no quasi-polynomial time algorithm that solves 3-SAT. So, finding a problem that can be solved in quasi-polynomial time and is also NP-hard, would contradict the ETH. In recent years, the ETH has become a standard assumption from which many conditional lower bounds have been proven [27]. Note that, in addition to violating the ETH, the existence of an NP-hard problem which can be solved in quasi-polynomial time would also directly imply that *nondeterministic exponential time (NEXP)* coincides with *deterministic exponential time (EXP)* (which can be proven by a padding argument similar to [20, Proposition 2]). Thus, having such an algorithm for a problem implies that it is extremely unlikely for that problem to be NP-hard.

1.3 Contribution

We first consider outer k-planar graphs; see Section 2. We show that the largest outer k-planar *complete* graph has $(\lfloor\sqrt{4k+1}\rfloor+2)$ vertices. Further, we show that every outer k-planar graph is $\lfloor 3.5\sqrt{k}\rfloor$-degenerate and hence has chromatic number at most $\lfloor 3.5\sqrt{k}\rfloor+1$. Next we show that every outer k-planar graph has separation number at most $2k+3$. While this does not improve the current best bound [33] of $k+2$ on the separator of outer k-planar graphs, we use our separator construction to obtain a quasi-polynomial time algorithm to test outer k-planarity (for fixed k), which means that the recognition problem is not NP-hard unless ETH fails. Very recently, this result has been improved by Kobayashi, Okada, and Wolff [46], who showed that the recognition problem can be solved in $2^{O(k\log k)}n^{3k+O(1)}$ time, that is, it lies in the class XP (slicewise polynomial). They also showed that the problem is XNLP-hard. This implies that it is W[t]-hard for every t and that it is unlikely that it admits a fixed-parameter algorithm, that is, an algorithm running in $f(k)\cdot n^{O(1)}$ time for some computable function f.

Then we show that the class of outer quasi-planar graphs and the class of planar graphs are incomparable; see Section 3.

Finally, we restrict outer k-planar and outer k-quasi-planar drawings to *full* drawings, i.e., drawings where no crossing appears on the boundary of the outer face, and to *closed* drawings, i.e., drawings where the vertex sequence on the boundary of the outer face is a cycle in the graph; see Section 4. (Note that every closed drawing is a full drawing.) The case of full outer 2-planar graphs has been considered by Hong and Nagamochi [41] who showed that full outer 2-planarity testing can be performed in linear time. They observed that a graph is full outer 2-planar if and only if its maximal biconnected components are closed outer 2-planar. We generalize this observation to k-planar and k-quasi-planar graphs, that is, a graph is full outer k-planar (full k-quasi-planar) if and only if its maximal biconnected components are closed outer k-planar (full k-quasi-planar). Then, for each k, we express *closed outer k-planarity* (and *closed outer k-quasi-planarity*) in *extended monadic second-order logic*. Thus, since outer k-planar graphs have bounded treewidth, full outer k-planar graphs can be recognized in $O(f(k)\cdot n)$ time, for a computable function f. In other words, this recognition problem *is* fixed-parameter tractable. We note that this result greatly generalizes the work of Hong and Nagamochi [41]. Our general approach via Courcelle's theorem is similar to that of Bannister and Eppstein [14] for computing the convex crossing number of a graph.

2 Outer *k*-planar graphs

In this section, we study the structural properties of outer k-planar graphs such as degeneracy and separation number. Based on these structural properties, we obtain bounds on the colorability of outer k-planar graphs as well as a quasi-polynomial-time recognition algorithm.

2.1 Degeneracy

First, we focus on the degeneracy of outer k-planar *complete* graphs. We show that the largest outer k-planar complete graph has $\lfloor\sqrt{4k+1}\rfloor+2$ vertices; see Lemma 1. This implies that there are outer k-planar graphs whose minimum degree is $\lfloor\sqrt{4k+1}\rfloor+1$. We then bound the degeneracy of general outer k-planar graphs by $\lfloor 3.5\sqrt{k}\rfloor$ from above; see Theorem 2.

▶ **Lemma 1.** *For every $k \geq 0$, the largest outer k-planar complete graph has $\lfloor\sqrt{4k+1}\rfloor+2$ vertices.*

Proof. The largest outerplanar complete graph is K_3, so the statement is correct for $k=0$.

Now let $k \geq 1$ and consider an outer k-planar drawing of K_n for some $n \geq 1$. Let e be an edge that splits the complete graph so that there are $(n-2)/2$ many vertices on both sides if n is even, or $(n-1)/2$ on one side and $(n-3)/2$ on the other if n is odd. Then the edge e has the largest number of crossings among all the edges in the drawing, namely $(n-2)^2/4 = (n^2-4n+4)/4$ if n is even and $(n-3)(n-1)/4 = (n^2-4n+3)/4$ if n is odd. Taking into account the fact that no edge is crossed more than k times, we obtain that $n \leq \lfloor 2\sqrt{k}\rfloor+2$ if n is even and $n \leq \lfloor\sqrt{4k+1}\rfloor+2$ if n is odd.

On the other hand, let $n = \lfloor 2\sqrt{k}\rfloor+2$. Place the vertices of K_n on the corners of a convex n-gon. Since $n \leq \lfloor\sqrt{4k+1}\rfloor+2$, it is clear that the resulting convex drawing is outer k-planar. We claim that this is even true for $n = \lfloor\sqrt{4k+1}\rfloor+2$.

Note that the two bounds deviate only if $4k+1$ is a square number, say ℓ^2. Then $\lfloor\sqrt{4k+1}\rfloor = \ell$, whereas $\lfloor\sqrt{4k}\rfloor = \lfloor\sqrt{\ell^2-1}\rfloor = \ell-1$. In this case, however, $n = \lfloor\sqrt{4k+1}\rfloor+2 = \ell+2$ is odd, because $4k+1$ is odd. Hence, the slightly larger bound applies, and K_n is outer k-planar. ◀

Let G be an outer k-planar graph. Consider some outer k-planar drawing of G. Recall that vertices lie on a circle and the edges are straight-line segments. We say that an edge ab *splits off* $l \in \mathbb{N}$ vertices of G *to one side* if one of the open half-planes defined by the edge ab contains exactly l vertices (not including a and b). From the context it will be clear which of the two half-planes we mean.

The following theorem gives an upper bound on the degeneracy of outer k-planar graphs.

▶ **Theorem 2.** *For every positive integer k, let δ_k be the degeneracy of outer k-planar graphs. Then $\delta_k \leq \lfloor c_k\sqrt{k}\rfloor$, where*

$$c_k = \frac{5+3\sqrt{1+8k}}{4\sqrt{k}}.$$

The sequence $(c_k)_{k\geq 1}$ is monotonically decreasing with $c_1 = 3.5$ and limit $3\sqrt{2}/2$.

Proof. Let G be an outer k-planar graph of minimum degree δ_k. Consider some outer k-planar drawing of G. Assume that there exists an edge e that splits off $t \in \mathbb{N}$ vertices in the drawing of G to one side, then there are at least $\delta_k t - t(t-1) - 2t = \delta_k t - t(t+1)$ edges crossing the edge e (on the left-hand side of the equality the second term stands for the sum of the degrees of a clique on t vertices and the third term for the number of edges incident to the endpoints of e and to the t many vertices). Because G is outer k-planar, we have that

$$\delta_k t - t(t+1) \leq k. \tag{1}$$

Therefore, either $t \leq t_1$ or $t_2 \leq t$, where

$$t_1 = \left((\delta_k-1) - \sqrt{(\delta_k-1)^2-4k}\right)/2 \quad\text{and}\quad t_2 = \left((\delta_k-1) + \sqrt{(\delta_k-1)^2-4k}\right)/2.$$

Assume for a contradiction that $\delta_k \geq c\sqrt{k}$ for some $c > c_k$, where c_k is as stated in the theorem. Then the solutions t_1 and t_2 ($t_1 < t_2$) to the quadratic equation (1) exist and are distinct, because $c_k > 1/\sqrt{k}+2$, and so, $\delta_k > c_k\sqrt{k} \geq 2\sqrt{k}+1$ and the discriminant of (1) is positive. Call an edge that splits off at least t_2 vertices to both sides *long*. The number ℓ of long edges incident to each vertex can be bounded from below by subtracting from δ^* the upper bound on the number of incident edges that are not long. Recall that those can split off at most t_1 vertices, so we can have at most $2(t_1+1)$ such edges incident to the same vertex. Thus we can bound ℓ from below as follows:

$$\begin{aligned}\ell &\geq \delta_k - 2(t_1+1)\\ &= \delta_k - 2\left(\left((\delta_k-1)-\sqrt{(\delta_k-1)^2-4k}\right)/2+1\right)\\ &= 1+\sqrt{(\delta_k-1)^2-4k}-2 \geq \sqrt{(c^2-4)k-2c\sqrt{k}+1}-1.\end{aligned}$$

Take a shortest long edge e' in the outer k-planar drawing of G, that is, in one of the open half-planes h defined by e', there is no long edge that is completely contained in h. Let V_h be the set of vertices of the graph that are contained in the half-plane h. Because e' is a long edge, $|V_h| \geq t_2 = \left((\delta_k-1)+\sqrt{(\delta_k-1)^2-4k}\right)/2$. Because it is a shortest long edge, all the long edges incident to the vertices in V_h must cross e'. Therefore, the number of long edges that cross e' is at least

$$\ell t_2 \geq \frac{1}{2}\left(\sqrt{(c^2-4)k-2c\sqrt{k}+1}-1\right)\left((c\sqrt{k}-1)+\sqrt{(c^2-4)k-2c\sqrt{k}+1}\right)$$

Let

$$f(c,k) = \frac{1}{2}\left(\sqrt{(c^2-4)k-2c\sqrt{k}+1}-1\right)\left((c\sqrt{k}-1)+\sqrt{(c^2-4)k-2c\sqrt{k}+1}\right)-k.$$

Consider the equation $f(c,k)=0$, where $k \in \mathbb{N}$ and $c \in (1/\sqrt{k}+2,\infty)$.

In order to find the roots of the above equation, we first observe that

$$(c^2-4)k-2c\sqrt{k}+1 = (c\sqrt{k}-1)^2-4k.$$

Therefore, $f(c,k) =$

$$\frac{1}{2}\left((c\sqrt{k}-1)^2-4k+(c\sqrt{k}-1)\sqrt{(c\sqrt{k}-1)^2-4k}-\sqrt{(c\sqrt{k}-1)^2-4k}-(c\sqrt{k}-1)\right)-k.$$

Let $X = c\sqrt{k}-1$, then the equation $f(c,k)=0$ becomes

$$X^2-4k+X\sqrt{X^2-4k}-\sqrt{X^2-4k}-X-2k=0$$

or

$$X^2-X-6k=(1-X)\sqrt{X^2-4k}$$

let us square it

$$X^4+X^2+36k^2-2X^3-12X^2k+12Xk=(1-x)^2(X^2-4k)$$

after expanding we have

$$-8X^2k+4Xk+36k^2+4k=0$$

or

$$X^2 - \frac{1}{2}X - \frac{9k+1}{2} = 0$$

the positive root

$$X = \frac{1 + 3\sqrt{1+8k}}{4}.$$

Then, due to our definition of X, we have

$$c = \frac{X+1}{\sqrt{k}} = \frac{5 + 3\sqrt{1+8k}}{4\sqrt{k}}.$$

Note that the right-hand side is actually c_k. This contradicts our assumption that $c > c_k$. Thus, for every $k \in \mathbb{N}$, the degeneracy of any outer k-planar graph is at most $c_k\sqrt{k}$. Observe that $(c_k)_{k\geq 1}$ is a monotonically decreasing sequence with $c_1 = 3.5$ and limit $3\sqrt{2}/2$. ◀

Lemma 1 implies that there are (complete) outer k-planar graphs with degeneracy $\lfloor 2\sqrt{k}\rfloor + 1$. Note that, for $k < 42$, our upper bound $\lfloor (5+3\sqrt{1+8k})/4\rfloor \approx 2.13\sqrt{k}$ from Theorem 2 differs only by 1 from the existential lower bound provided by Lemma 1.

As a direct consequence of Theorem 2, we obtain the following.

▶ **Corollary 3.** *Every outer k-planar graph has at most $\lfloor c_k\sqrt{k}\rfloor n$ edges, where n is the number of vertices.*

For small k (recall that $c_1 = 3.5$) there is a stronger (and more general) bound by Aichholzer, Obenaus, Orthaber, Paul, Schnider, Steiner, Taubner, and Vogtenhuber [6] of roughly $2.46\sqrt{k}n$ for the maximum edge density of outer k-planar graphs. However, as k tends to infinity, our upper bound $(5 + 3\sqrt{1+8k})/4 \cdot n$ tends to $(3\sqrt{2}/2)\sqrt{k}n \approx 2.13\sqrt{k}n$. Hence, our bound is less than the bound by Aichholzer et al. for $k \geq 15$ (ignoring the floors).

By combining Lemma 1 and Theorem 2, we obtain the following.

▶ **Corollary 4.** *Every outer k-planar graph can be colored with $\lfloor c_k\sqrt{k}\rfloor + 1$ colors. There exist outer k-planar graphs that need at least $\lfloor\sqrt{4k+1}\rfloor + 2$ colors.*

2.2 Quasi-polynomial-time recognition via balanced separators

We now show that outer k-planar graphs have separation number at most $2k+3$ (Theorem 5). Via a result of Dvořák and Norin [32], this implies that their treewidth is at most $30k + 45$. However, a result of Wood and Telle [61, Proposition 8.5] implies that every outer k-planar graph has treewidth at most $3k + 11$, which is a better bound than what we get by applying the result of Dvořák and Norin to our separators. The treewidth bound of $3k + 11$ in turn implies a separation number of $3k + 12$, but our bound is better. Our separators also allow us to test outer k-planarity in quasi-polynomial time; see Theorem 9.

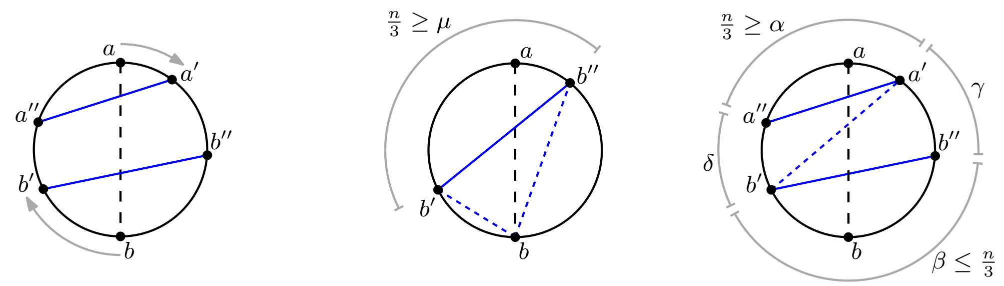
Very recently, both the result of Wood and Telle and our result have been improved by Firman, Gutowski, Kryven, Okada, and Wolff [33], who showed that outer k-planar graphs have treewidth at most $\lfloor 1.5k\rfloor + 2$ and separation number at most $k + 2$. They also show that $k + 2$ is an existential lower bound for both numbers.

▶ **Theorem 5.** *Each outer k-planar graph has separation number at most $2k + 3$.*

Proof. Consider an outer k-planar drawing. If the graph contains an edge e that splits off at least $n/3$ and at most $2n/3$ vertices to one side, we can use e to obtain a balanced separator of size at most $k+2$: take the endpoints of e and a vertex cover of the edges that cross e. Thus, in the following we assume that no such edge exists.

Consider a pair of vertices a and b such that the *line* through ab divides the drawing into left and right sides having an almost equal number of vertices (with a difference of at most one). If the edges which cross the line ab also mutually cross each other, there can be at most k of them. Thus, we again have a balanced separator of size at most $k+2$. So assume now that there is a pair of edges that cross the line ab, but do not cross each other. We call such a pair of edges *parallel*. We now pick a pair of parallel edges in a specific way. Starting from b, let b' be the first vertex along the boundary in clockwise direction such that there is an edge $b'b''$ that crosses the line ab. Symmetrically, starting from a, let a' be the first vertex along the boundary in clockwise direction such that there is an edge $a'a''$ that crosses the line ab; see Fig. 2a. Note that the edges $a'a''$ and $b'b''$ are either identical or parallel. In the former case, we see that all other edges crossing the line ab must also cross the edge $a'a'' = b'b''$, and hence there are at most k edges crossing the line ab. In the latter case, there are two subcases that we treat below.

For two vertices u and v, let $[u,v]$ be the set of vertices that starts with u and, going clockwise, ends with v. Let $(u,v) = [u,v] \setminus \{u,v\}$. When we say that an edge uv splits off α vertices without specifying the side, we mean that $|[u,v]| = \alpha$. Note that this implies that the edge vu splits off $n - \alpha + 2$ vertices.



(a) edges $a'a''$ and $b'b''$ are parallel **(b)** case 1 **(c)** case 2

Figure 2 Illustration for the proof of Theorem 5

Because we assumed that there is no edge that splits off at least $n/3$ and at most $2n/3$ vertices, our case distinction below is exhaustive.

Case 1. *The edge $b'b''$ splits off at most $\mu \le n/3$ vertices (see Fig. 2b).*
In this case, $n/3 \le |[b,b']| \le n/2$ or $n/3 \le |[b'',b]| \le n/2$. We claim that neither the line bb' nor the line bb'' is crossed more than k times. Namely, due to the choice of b', each edge that crosses the line bb' also crosses the edge $b'b''$. Also due to the choice of b', each edge that crosses the line bb'' also crosses the edge $b'b''$. Thus, we have a separator of size at most $k+2$, regardless of whether we choose bb' or bb'' to separate the graph. As we observed above, one of them is balanced.

Case 1′. *The edge $a'a''$ splits off at most $n/3$ vertices.*
This is symmetric to case 1.

Case 2. *The edge $a'a''$ and the edge $b'b''$ both split off at most $n/3$ vertices (see Fig. 2c).*
We show that we can even find such a pair of edges $a'a''$ and $b'b''$ with the additional requirement that there is no edge cd between them that is parallel to them, that is, $c \in [a', b'']$

and $d \in [b', a'']$ and $cd \notin \{a'a'', b'b''\}$. We call such a pair *close*. If there is an edge $e = uv$ between $a''a'$ and $b'b''$, we form a new pair by using uv and $a'a''$ if uv splits off at most $n/3$ vertices or by using uv and $b'b''$ if vu splits off at most $n/3$ vertices. Note that one of the two conditions must hold due to our assumption that there is no edge that splits off at least $n/3$ and at most $2n/3$ vertices. By repeating this procedure, we always find a close pair. Hence, we can assume that $a'a''$ and $b'b''$ actually form a close pair. Let $\alpha = |(a'', a')|$, $\beta = |(b'', b')|$, $\gamma = |(a', b'')|$, and $\delta = |(b', a'')|$; see Fig. 2c. Note that $n - 4 \leq \alpha + \beta + \gamma + \delta \leq n - 3$.

We further consider two cases depending on whether the two edges $a''a'$ and $b''b'$ share an endpoint or not.

First, suppose that $a' = b''$ or $a'' = b'$ (that is, $\gamma = 0$ or $\delta = 0$). We can now use both edges $a''a'$ and $b''b'$ (together with any edges crossing them) to obtain a separator of size at most $2k + 3$. The separator is balanced since $\alpha + \beta \leq 2n/3$ and $\gamma + \delta \leq n/2$. The latter holds due to the fact that $\gamma = 0$ or $\delta = 0$ and that $\gamma \leq n/2$ and $\delta \leq n/2$ since, on each side of the line ab, there are at most $n/2$ vertices.

Second, assume that a', a'', b', b'' are all distinct. Note that we again have that $\gamma \leq n/2$ and $\delta \leq n/2$. We separate the graph along the line $a'b'$; see Fig. 2c. Namely, all the edges that cross this line must also cross the edge $a''a'$ or the edge $b''b'$. Therefore, we obtain a separator of size at most $2k + 2$.

To see that the separator is balanced, we consider two cases. If $\delta \geq n/3$ (or $\gamma \geq n/3$), then $\alpha + \beta + \gamma \leq 2n/3$ (or $\alpha + \beta + \delta \leq 2n/3$). Otherwise $\delta < n/3$ and $\gamma < n/3$. In this case $\delta + \alpha \leq 2n/3$ and $\gamma + \beta \leq 2n/3$. In both cases the separator is balanced. ◀

Let G be an outer k-planar graph and consider some outer k-planar drawing D of G. According to Theorem 5, G has a balanced separator of size at most $2k+3$ that arises from D as described in Theorem 5. Furthermore, we observe the following structure of the separator that follows from the proof of Theorem 5.

▶ **Observation 6.** *Each outer k-planar graph with an outer k-planar drawing D has a balanced separator of size at most $2k + 3$ that is of one of the following two types:*

(1) *three vertices x, y, and z that are endpoints of two close parallel edges xy and yz and some endpoints of the edges crossing the edges xy and yz in D and*

(2) *two vertices x and y and some endpoints of the edges crossing the line xy in D.*

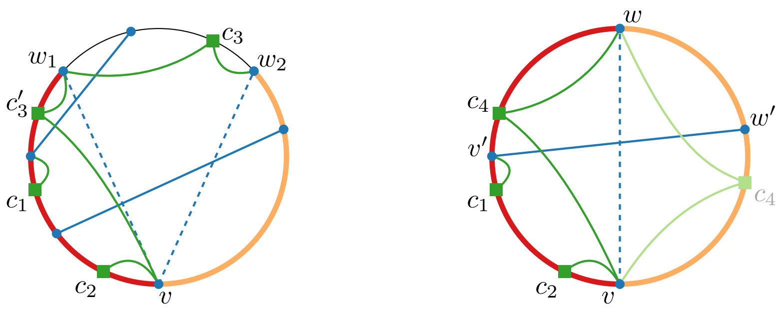
We call the vertices x, y, and z *boundary vertices*.

Next we show that we can recognize whether a given graph G is outer k-planar in quasi-polynomial time; see Theorem 9. Our proof relies on Theorem 5; in particular, it uses the structure of balanced separators as described in Observation 6.

We need the following notation. Given an outer k-planar drawing D of G, let S be a separator constructed as in the proof in Theorem 5, and let $S' \subset S$ be the set of boundary vertices. We call each arc of the circle of the drawing D that connects two consecutive vertices of S' a *region* with respect to S'. We call the vertices in $S \setminus S'$ *regional* vertices. Note that if the separator is of type (1), then there are three regions, and if it is of type (2), then there are two regions.

▶ **Lemma 7.** *For each connected component of $G \setminus S$, its vertices are only in one region of D with respect to S'.*

Proof. Assume for a contradiction that there is a connected component with an edge e with endpoints in two different regions. Then e crosses a line between two boundary vertices in D. Therefore, e is a separator edge and at least one of its endpoints lies in S. ◀



(a) a separator of type (1) **(b)** a separator of type (2)

Figure 3 The two types of separators according to Observation 6.

We also have the following.

▶ **Lemma 8.** *If S is a type-(1) separator, then no connected component of $G \setminus S$ is adjacent to all three boundary vertices.*

Proof. Assume for a contradiction that there is a connected component of $G \setminus S$ connected by edges in $E(G) \setminus E(G \setminus S)$ to each of the boundary vertices x, y, and z. If the component lies in the region bounded by x and z, then the edges xy and yz are not a close parallel pair in D. This yields the desired contradiction. If the component lies in one of the regions bounded by x and y or by y and z, an edge e of the component crosses either the line xy or the line yz in D. Again, this yields the desired contradiction since then e is one of the separator edges. ◀

In the remainder of this section, we refer to the connected components of $G \setminus S$ simply as *components*.

In order to obtain a quasi-polynomial algorithm for recognizing outer k-planar graphs, we leverage the structure of balanced separators as described in the proof of Theorem 5. Specifically, we enumerate the vertex sets that could form such a separator. For each set, we choose an appropriate outer k-planar drawing of the subgraph it induces and partition the remaining parts of the graph into regions bounded by the boundary vertices of the separator. We then recursively test whether the graph induced by the vertices in each of the regions admits an outer k-planar drawing.

▶ **Theorem 9.** *For every fixed k, testing whether a given graph is outer k-planar takes $O(2^{\mathrm{polylog}(n)})$ time, where n is the number of vertices of the given graph.*

Proof. To obtain quasi-polynomial runtime, we need to limit the number of regions on which we branch. Let G be the given graph. According to Theorem 5, if G is outer k-planar, then there exists a set S with at most $4k+3$ vertices where the vertices are endpoints of the edges of a balanced separator. Note that we are interested in all the endpoints of the edges, not just those that form the separator. In addition, note that S is also a balanced separator. Furthermore, S is of one of the two types described in Observation 6; see Fig. 3a for a type-(1) separator and Fig. 3b for a type-(2) separator. We find S by brute force, enumerating all vertex sets of size at most $4k+3$. Once S is fixed, observe that the subgraph $G[S]$ induced by S admits only $f(k)$ possible outer k-planar drawings, for some function f. We then enumerate all such possible drawings. Assume now that one such drawing of S is fixed.

We now consider two cases depending on the type of the separator S. If S is of type (1), then S contains three boundary vertices v, w_1, and w_2; see Fig. 3a. If S is of type (2), then there are two boundary vertices v and w; see Fig. 3b. Note that, since we have a fixed drawing of $G[S]$, the regional vertices are partitioned into regions as defined by the boundary vertices of S. According to Lemma 7, no component of $G \setminus S$ is connected to regional vertices of different regions.

Separator of type (1) First, assume that the separator S is of type (1); see Fig. 3a. We start by choosing the three boundary vertices v, w_1, and w_2 from S. According to Lemma 7, if there is a component connected to regional vertices of different regions, we can reject this configuration. Furthermore, according to Lemma 8, no component is adjacent to all three boundary vertices. We now consider three possible different types c_1, c_2, and c_3 of components depending on the type of the vertices (boundary or regional) that they are connected to; see Fig. 3a.

Components of type c_1 are connected to (possibly many) regional vertices of the same region and may be connected to boundary vertices as well. In any valid drawing, they will end up in the same region as their regional vertices.

Components of type c_2 are not connected to any regional vertices; instead, they are connected to exactly one of the boundary vertices. Since they are not connected to regional vertices, they cannot interfere with other parts of the drawing, so we can arbitrarily assign them to a region adjacent to the boundary vertex they are connected to.

Finally, there are components that are connected to two boundary vertices. There are two types of such components.

The first type (which we call c_3) consists of those components that are connected to w_1 and w_2. These have to be placed in the region bounded by w_1 and w_2. Otherwise one of the edges connecting such a component would cross the line vw_1 or the line vw_2, and thus, it would be a separator edge and therefore it could not be part of the component.

The second type (which we call c_3') consists of those components that are either connected to v and w_1 or to v and w_2. These have to be placed in the region bounded by v and w_1 or by v and w_2, respectively, since otherwise (similarly as above) one of the edges connecting such a component would cross the line vw_1 or the line vw_2 and thus, it would be a separator edge and could not be part of the component. Note that a component connected to v and w_1 or to v and w_2 cannot be placed in the region between w_1 and w_2 since this would contradict the fact that our separator arose from a pair of close parallel edges as argued in the proof of Theorem 5.

The above discussion implies the following for our recognition algorithm. If, in a fixed configuration (i.e., set S, drawing of $G[S]$, and triplet of boundary vertices), the drawing of $G[S]$ corresponds to a separator of type (1), then we can reject the current configuration (based on having components that do not correspond to any of the types c_1, c_2, c_3 or c_3') – or every component of $G \setminus S$ is either attached to exactly one boundary vertex or it has a well-defined placement into a region defined by its adjacent boundary vertices. For each component that is attached to exactly one boundary vertex, it suffices to recursively produce a drawing of that component together with its boundary vertex and to place this drawing next to the boundary vertex. We place the other components into their regions and recurse on the regions. This covers all cases for separators of type (1).

Separator of type (2) Second, assume that the separator S is of type (2); see Fig. 3b. Note that we now have two boundary vertices v and w; thus we have only two regions. Similarly as above, we now consider three possible different types c_1, c_2, and c_4 of components depending on the type of vertices (boundary or regional) that they are connected to; see Fig. 3b.

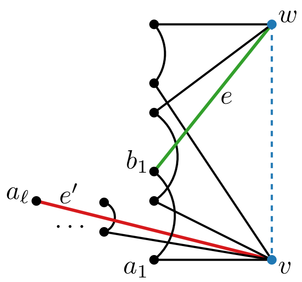


Figure 4 If the pair $\{v, w\}$ is a separator, we group the components of $G - \{v, w\}$ into layers.

Components c_1 and c_2 are defined exactly as they were defined for separators of type (1). We handle these components as described above.

A separator of type (2) may, however, also create components that are connected to both boundary vertices v and w, but to no regional vertices; we refer to these as components of type c_4. Such components can be placed in either of the two regions.

If $G[S]$ contains an edge $v'w'$ that crosses the line vw (as in Fig. 3b), then there cannot be more than k components of type c_4. Namely, in any drawing, every type-c_4 component contains an edge that connects the component to either v or w and that hence crosses $v'w'$. Thus, we enumerate all the different placements of these components and recurse accordingly.

On the other hand, if $G[S]$ does not contain any edge that crosses the line vw, then the separator is exactly the pair $\{v, w\}$. In this case, there are no components of type c_1. Components of type c_2 can be handled as before. We now argue that we can have at most a function of k many different components of type c_4 in a valid drawing. Consider the components of type c_4. In a valid drawing, each type-c_4 component defines an interval of the region from its *highest* vertex (that is, the vertex closest to w) that is adjacent to v or w to its *lowest* vertex (that is, the vertex closest to v) that is adjacent to v or w. Two such intervals relate in one of three ways: they overlap each other, they are disjoint, or one contains the other. We group components with either overlapping or disjoint intervals into *layers*. We depict this situation in Fig. 4 where, for simplicity, we draw only the highest and the lowest vertex of each component, and we represent the path between them by a single edge in the drawing.

Let a_1b_1 be the interval of the bottommost type-c_4 component (i.e., going from v in clockwise direction, a_1 is the first vertex adjacent to v or w in a type-c_4 component). The first layer consists of this component and every component whose interval either overlaps or is disjoint from the interval a_1b_1. We now consider the edge $e \in \{b_1w, b_1v\}$ (whichever exists; drawn green in Fig. 4). For every component of the first layer whose interval is disjoint from a_1b_1, the edge that connects the component to v crosses e. For every component of the first layer whose interval overlaps a_1b_1, the edge e is crossed by at least one edge within that component. Hence, the first layer consists of at most $k + 1$ components. The next layer consists of all components whose intervals are contained in the intervals of components on the first level etc. Let ℓ be the index of the deepest layer, let a_ℓ be the bottommost vertex of that layer connected to v or w. If e' is the edge that connects the component of a_ℓ to v (red in Fig. 4), then e' is crossed by some edge of every layer above it. Hence there are at most $k + 1$ levels and at most $(k + 1)^2$ components per region. This means that we can enumerate their possible placements and recurse accordingly.

We now bound the runtime of the above recursive algorithm. Let $T(n)$ denote its worst-case runtime when applied to an outer k-planar graph with n vertices. Then,

$$T(n) = \begin{cases} n^{O(k)} \cdot f(4k+3) \cdot g((k+1)^2) \cdot n^4 \cdot T(2n/3) & \text{for } n > 5k, \\ f(n) & \text{otherwise,} \end{cases}$$

where $f(s)$ denotes the number of different outer k-planar drawings of a graph with s vertices and $g(s)$ denotes the number of ways s components can be partitioned into at most three regions. The factor $n^{O(k)}$ stands for finding all possible separators of size $4k+3$, and $T(2n/3)$ is the runtime of the recursive call on a region.

Thus, the algorithm runs in quasi-polynomial time, i.e., $T \in 2^{\text{polylog}(n)}$. ◀

3 Outer quasi-planar graphs

In this section we consider outer quasi-planar graphs, that is, outer 3-quasi-planar graphs. We describe some classes of graphs that are outer quasi-planar and some classes of graphs that are not outer quasi-planar. In particular, we show that there are planar graphs that are not outer quasi-planar. Hence, the classes of planar graphs and outer quasi-planar graphs are incomparable; see Theorem 12.

A graph is *sub-Hamiltonian* if it is a subgraph of a Hamiltonian graph. A *planar 3-tree* is either K_4 or a graph G formed by *stacking* a vertex into a smaller planar 3-tree G', that is, by inserting a vertex into a triangular face of G' and by connecting the new vertex to the three vertices of the triangular face of G'. The graph K_4 is the only complete planar 3-tree of level 1. For $i > 1$, the *complete planar 3-tree of level i* is obtained from the complete planar 3-tree of level $i-1$ by stacking a vertex into every triangular face of the complete tree of level $i-1$.

▶ **Proposition 10.** *The following graphs are outer quasi-planar: (a) every sub-Hamiltonian planar graph; (b) $K_{p,q}$ for $p \leq 4$ and $q \leq 4$; (c) K_n for $n \leq 5$; (d) the complete planar 3-tree of level at most 3; (e) grid graphs of any size.*

Proof. Clearly, every subgraph of an outer quasi-planar graph is outer quasi-planar.

To see (a), consider a planar drawing of the given graph G. It partitions the edge set of G into the set E_0 edges that lie on the cycle, the set E_1 of edges that lie in the interior of the cycle, and the set E_2 of edges that lie in the exterior of the cycle. Now place the vertices of G on a circle so that their order follows the Hamiltonian cycle. The induced straight-line drawing of G is outer quasi-planar since the edges in E_0 do not cross any edges, no two edges in E_1 cross, and no two edges in E_2 cross. Hence, every set of three edges can have at most two crossings.

For (b) and (c), note that the two drawings of K_5 and $K_{4,4}$ in Fig. 5 are outer quasi-planar. (For K_5 this is trivial, because K_5 does not contain three independent edges.)

(a) K_5 **(b)** $K_{4,4}$

Figure 5 Drawings showing that K_5 and $K_{4,4}$ are outer quasi-planar.

We have verified claim (d) by constructing a SAT formulation; see Appendix A.1.

Claim (e) follows from the fact that grid graphs are sub-Hamiltonian. Note that grid graphs of odd order are not Hamiltonian, but they are subgraphs of larger grids of even order, which are Hamiltonian. Therefore, all grids are sub-Hamiltonian. ◀

Below, we identify complete and complete bipartite graphs that are not outer quasi-planar. Furthermore, not all planar graphs are outer quasi-planar, e.g., Fig. 6(a) shows a planar 3-tree that is not outer quasi-planar (but removing any vertex renders it outer quasi-planar). This was verified using a SAT formulation; see Appendix A.1. The drawing of the graph in Fig. 6(b) was constructed by removing the blue vertex and drawing the remaining graph in an outer quasi-planar way.

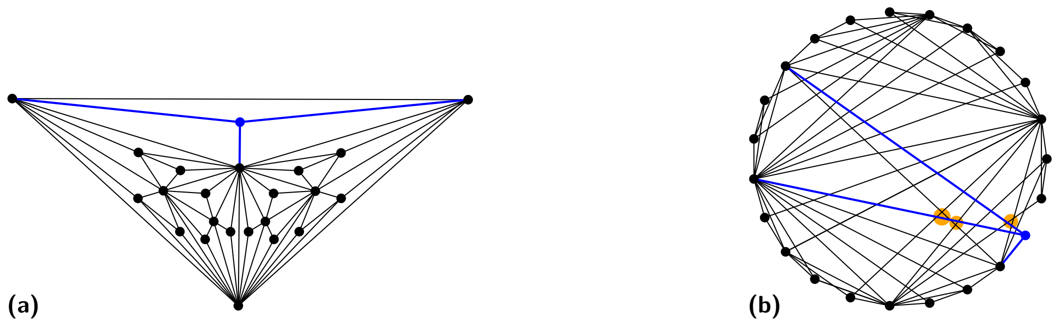
▶ **Proposition 11.** *The following graphs are not outer quasi-planar: (a) $K_{p,q}$, for $p \geq 3$ and $q \geq 5$; (b) K_n, for $n \geq 6$; (c) every planar 3-tree with at least four complete levels.*

Proof. Using the SAT formulation in Appendix A.1, we verified that $K_{3,5}$, K_6, and the planar 3-tree with four complete levels are all not outer quasi-planar. Clearly, every graph that contains any of these three graphs as subgraphs is also not outer quasi-planar. ◀

Together, Propositions 10 and 11 immediately yield the following.

▶ **Theorem 12.** *The class of planar graphs and the class of outer quasi-planar graphs are incomparable under containment.*

▶ Remark 13. For outer k-quasi-planar graphs with $k > 3$, containment questions become more intricate. Every planar graph is outer 5-quasi-planar because planar graphs have page number 4 [62]. There are planar graphs that are not outer quasi-planar (the planar 3-trees with at least four complete levels). It is open whether every planar graph is outer 4-quasi-planar.



(a) (b)

Figure 6 A planar 3-tree G (with 23 vertices) that is not outer quasi-planar: (a) planar drawing of G; (b) convex drawing of G that is not outer quasi-planar (see the highlighted triangular inner faces) but contains an outer quasi-planar drawing of G minus the blue vertex.

4 Testing for full convex drawings via MSO_2

The class of *full outer k-planar graphs* was introduced by Hong and Nagamochi [41]. Recall that this class consists of the graphs that admit a k-planar convex drawing where no crossing lies on the boundary of the outer face. Hong and Nagamochi gave a linear-time recognition algorithm for full outer 2-planar graphs. They state that a graph G is (full) outer 2-planar if and only if its biconnected components are (full) outer 2-planar and that the outer boundary of a full outer 2-planar drawing of a biconnected graph G is a Hamiltonian cycle of G. We call the subclasses of outer k-planar and outer k-quasi-planar graphs that have a convex drawing where the circular order forms a Hamiltonian cycle *closed outer k-planar* and *closed outer k-quasi-planar*, respectively. We observe that the property stated by Hong and Nagamochi carries over to general outer k-planar and outer k-quasi-planar graphs.

▶ **Observation 14** ([41]). *A graph G is full outer k-planar (full outer k-quasi-planar) if and only if its biconnected components are closed outer k-planar (closed outer k-quasi-planar).*

This observation follows from the fact that each biconnected component of a full outer k-planar (full outer k-quasi-planar) drawing is bounded by a cycle, and thus, is a closed outer k-planar (closed outer k-quasi-planar) graph.

We show that we can encode closed outer k-planarity and closed outer k-quasi-planarity using Monadic Second-Order Logic (MSO_2). To do so, we initially give a brief introduction to MSO_2 and Courcelle's theorem. Then we design MSO_2 formulas expressing crossing patterns of closed k-planar and closed k-quasi-planar drawings. Using Observation 14, we can test the full outer k-planarity of a graph by testing its biconnected components for closed outer k-planarity using the MSO_2 formulas. This, together with Courcelle's theorem (see Theorem 15 below) and the fact that outer k-planar graphs have bounded treewidth (see [61, Proposition 8.5] or [33]) yields a linear-time algorithm for testing full outer k-planarity. Unfortunately, grid graphs, which are outer quasi-planar, have unbounded treewidth. So the same approach does not yield a linear-time algorithm for testing full outer k-quasi-planarity.

Monadic Second-Order Logic (MSO_2) – a subset of *second-order logic* – can be used to express certain graph properties. It is built from the following primitives.

- variables for vertices, edges, sets of vertices, and sets of edges;
- binary relations for: equality ($=$), membership in a set ($\in$), subset of a set ($\subseteq$), and edge–vertex incidence (I);
- standard propositional logic operators: $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$.
- standard quantifiers ($\forall, \exists$) which can be applied to all types of variables.

Properties expressed in this logic allow us to use the powerful algorithmic result of Courcelle stated next.

▶ **Theorem 15** ([25, 26]). *For any integer $t \geq 0$ and any MSO_2 formula ϕ of length ℓ, an algorithm can be constructed that takes a graph G and decides whether G satisfies ϕ. If G has n vertices, m edges, and treewidth at most t, then the algorithm runs in time $O(f(t, \ell) \cdot (n + m))$, where the function f is computable.*

The challenge in expressing outer k-planarity or outer k-quasi-planarity in MSO_2 is that MSO_2 does not allow quantification over sets of pairs of vertices which involve non-edges. Namely, it is unclear how to express a set of pairs that forms the circular order of vertices on the boundary of our convex drawing. However, if this circular order forms a Hamiltonian cycle in our graph, i.e., the given graph is closed, then we can indeed express this in MSO_2. With the edge set of a Hamiltonian cycle of our graph in hand, we can then ask that this cycle was chosen in such a way that the other edges satisfy either k-planarity or k-quasi-planarity.

The MSO_2 formulas presented below assume that a graph G is given and uses the edges, vertices, and incidences of G. In the following, V is the vertex set of G, $u, v \in V$, and $U \subseteq V$. Further, E is the edge set of G, $e, f \in E$, and $F \subseteq E$. (We also use sub- and superscripted variants of these variables.) In addition to the quantifiers above, we use $\exists^{=x}$ as shorthand to express the existence of exactly x (but no $x + 1$) pairwise different elements satisfying a property.

The first formula expresses the connectivity of a subgraph induced by a given edge set F.

$$\textsc{Connected-Edges}(F) \equiv (\forall F')\big([F' \neq \emptyset \wedge F' \subsetneq F] \rightarrow (\exists e, f, v)[e \in F' \wedge f \in F \setminus F' \wedge v \in V \wedge I(e,v) \wedge I(f,v)]\big)$$

It states that, for every nonempty proper subset F' of the given edge set F, we can find an edge e in F', an edge f in $F \setminus F'$, and a vertex that is incident to both.

We use the predicate Connected-Edges(F) and the following predicates to express the Hamiltonicity of G. The predicate Cycle-Set(F) expresses that F forms a set of cycles, Cycle(F) expresses that F consists of a single cycle, and Span(F) forces F to span G.

$$\begin{aligned}
\textsc{Cycle-Set}(F) &\equiv (\forall e)\Big[e \in F \rightarrow (\exists^{=2} f)\big[f \in F \wedge e \neq f \wedge (\exists v)[I(e,v) \wedge I(f,v)]\big]\Big]\\
\textsc{Cycle}(F) &\equiv \textsc{Cycle-Set}(F) \wedge \textsc{Connected-Edges}(F)\\
\textsc{Span}(F) &\equiv (\forall v)(\exists e)[e \in F \wedge I(e,v)]\\
\textsc{Hamiltonian}(F) &\equiv [\textsc{Cycle}(F) \wedge \textsc{Span}(F)]
\end{aligned}$$

The following predicate Vertex-Partition expresses the existence of a partition of a set U of vertices into k disjoint subsets.

$$\textsc{Vertex-Partition}(U, U_1, \ldots, U_k) \equiv (\forall v \in U)\left[\left(\bigvee_{i=1}^{k} v \in U_k\right) \wedge \left(\bigwedge_{i \neq j} \neg(v \in U_i \wedge v \in U_j)\right)\right]$$

Using Vertex-Partition, we can formulate

$$\begin{aligned}
\textsc{Connected}(F, U) \equiv (\forall U_1, U_2 \subseteq U)\Big(&\textsc{Vertex-Partition}(U, U_1, U_2)\\
&\wedge \big(\exists e \in F, u_1 \in U_1, u_2 \in U_2\big)\big(I(e,u_1) \wedge I(e,u_2)\big)\Big),
\end{aligned}$$

which is true if and only if the given set U of vertices is connected via the given set F of edges.

For a closed outer k-planar or closed outer k-quasi-planar graph G, we want to express that two edges e and e_i cross. To this end, we assume that G contains a Hamiltonian cycle E^* that defines the outer face. We partition the vertices of G into three subsets C, A, and B, as follows: C is the set containing the endpoints of e, whereas A and B are connected subgraphs on the remaining vertices that use only edges of E^*. In this way, we partition the vertices of G into two sets, one left of e and one right of e. For such a partition, e_i must cross e whenever e_i has one endpoint in A and one in B.

$$\begin{aligned}
\textsc{Crossing}(E^*, e, e_i) \equiv (\forall A, B, C)\big[\big(&\textsc{Vertex-Partition}(V, A, B, C)\\
&\wedge (I(e,x) \leftrightarrow x \in C) \wedge \textsc{Connected}(A, E^*) \wedge \textsc{Connected}(B, E^*)\big)\\
&\rightarrow (\exists a \in A)(\exists b \in B)[I(e_i, a) \wedge I(e_i, b)]\big]
\end{aligned}$$

Now we can describe the allowed crossing patterns. We first express closed outer k-planarity:

$$\textsc{Closed Outer } k\textsc{-Planar}_G \equiv (\exists E^*)\Big[\textsc{Hamiltonian}(E^*) \wedge$$

$$(\forall e)\Big[(\forall e_1, \ldots, e_{k+1})\Big[\Big(\bigwedge_{i=1}^{k+1} e_i \neq e \wedge \bigwedge_{i \neq j} e_i \neq e_j\Big) \rightarrow \bigvee_{i=1}^{k+1} \neg\textsc{Crossing}(E^*, e, e_i)\Big]\Big]\Big]$$

Here we insist that G is Hamiltonian and that, for every edge e and any set of $k+1$ distinct other edges, at least one among them does not cross e.

Now we express closed outer k-quasi-planarity:

$$\textsc{Closed Outer } k\textsc{-Quasi-Planar}_G \equiv (\exists E^*)\Big[\textsc{Hamiltonian}(E^*) \wedge$$

$$(\forall e_1, \ldots, e_k)\Big[\Big(\bigwedge_{i \neq j} e_i \neq e_j\Big) \rightarrow \bigvee_{i \neq j} \neg\textsc{Crossing}(E^*, e_i, e_j)\Big]\Big]$$

Again, we insist that G is Hamiltonian and further that, for any set of k distinct edges, there is at least one pair among them that does not cross.

The formulas above give us the following.

▶ **Theorem 16.** *Closed outer k-planarity and closed outer k-quasi-planarity can be expressed in MSO_2 with a formula whose size depends only on k.*

Using the fact that outer k-planar graphs have bounded treewidth yields the following recognition result.

▶ **Theorem 17.** *We can test whether a graph is full outer k-planar in time linear in the size of the graph.*

Proof. Recall that in a full outer k-planar drawing there is no crossing on the outer boundary of the drawing and each biconnected component of the graph with such a drawing is a closed outer k-planar graph (Observation 14). Thus, in order to test full outer k-planarity of a given graph G, it suffices to test whether each of its biconnected components admits a closed outer k-planar drawing. We can break up G into biconnected components by obtaining the set of cutvertices [42]. This takes linear time. Checking each biconnected component for closed outer k-planarity can be done via the above MSO_2 formula in time linear in the size of the component; see Theorems 15 and 16. The formula also guarantees that the Hamiltonian cycle (if present) is placed on the outer boundary of the drawing of each component. We put the individual drawings of the components together by reidentifying the cutvertices and without introducing any crossings. This can also be done in linear time. Thus, the total runtime is linear in the size of the input graph G. ◀

5 Discussions and open problems

Every planar graph is outer 5-quasi-planar because planar graphs have page number 4 [62]. (Planar graphs that require four pages have been discovered recently [15,63]). There are also planar graphs that are not outer quasi-planar. The following question is still open.

▶ **Question 18.** *Is every planar graph outer 4-quasi-planar?*

We now discuss the relation between our crossing-restricted convex drawings and the class of intersection graphs of chords of a circle, i.e., *circle graphs*. Such representations are called *chord diagrams*. Here, a convex drawing D of a graph G can be seen as a chord diagram and as such provides a corresponding graph H where each adjacency between two vertices corresponds to a crossing between the edges of our drawing. Independent sets in H correspond to collections of pairwise non-crossing edges in D, i.e., outerplanar sub-drawings of D. Thus, k-coloring H corresponds to partitioning D into edge sets $E_1, \ldots, E_k$ such that each sub-drawing of D formed by the edges of E_i is outerplanar. That is, the partition $E_1, \ldots, E_k$ forms a book embedding of G with k pages. So, k-coloring the chord diagram provides a k-page book embedding of G. Interestingly, it is NP-complete to test whether a chord diagram can be 4-colored [36], but testing whether it can be 3-colored is still open [60]. On the other hand, circle graphs are *χ-bounded* [48], i.e., the chromatic number $\chi(G)$ of a circle graph G is bounded by a function of the *clique number* $\omega(G)$ of G, that is, the number of vertices in the maximum clique of G. Until recently the best known bound was $7\omega^2$ due to Davis et al. [28], but then Davis showed [29] an improved bound of $O(\omega \log \omega)$, which is asymptotically tight. In particular, this means that every outer k-quasi-planar drawing can be partitioned into $O(k \log k)$ pages (since we cannot have k mutually crossing edges, i.e., there is no k-clique in the corresponding intersection graph). For quasi-planar graphs ($k = 3$) there is a tighter bound. Ageev [4] showed that any triangle-free circle graph has chromatic number at most 5. Because for a fixed drawing of an outer quasi-planar graph its corresponding circle graph is triangle-free, it has chromatic number at most 5, and thus, we can embed the outer quasi-planar graph in a book with five pages. An immediate open question is to improve this bound on the page number.

Ageev [4] constructed a triangle-free circle graph G_{ageev} with $\chi(G_{\text{ageev}}) = 5$. The drawing of the outer quasi-planar graph G corresponding to the circle graph G_{ageev} cannot be embedded on four pages because the circle graph has chromatic number 5. It turns out, however, that there exists a linear order of the vertices under which G can be embedded on four pages, even if we add edges to make it maximal, but there does not exist such an order with the additional property that the drawing is outer quasi-planar. We have verified this by constructing a logical formula that tests outer quasi-planarity (Appendix A.1) and 4-page embeddability (Appendix A.2) at the same time.

▶ **Question 19.** *Does every outer quasi-planar graph have page number at most 4?*

We also point to the gap between the lower bound (see Lemma 1) and the upper bound (see Theorem 2) for the degeneracy of outer k-planar graphs.

▶ **Question 20.** *Can we improve the lower bound of $\lfloor\sqrt{4k+1}\rfloor + 1$ or the upper bound of $\lfloor 3.5\sqrt{k}\rfloor$ on the degeneracy of outer k-planar graphs?*

The linear runtime algorithm for testing full outer k-planarity in Theorem 17 relies on Courcelle's machinery for solving MSO_2 formulas and, therefore, has a notoriously bad runtime dependence (of $2^{O(k^2)}$) on the parameter k. Hence, in light of a similar result for one-page crossing minimization [45], it is natural to ask whether this can be improved.

▶ **Question 21.** *Is there an explicit dynamic programming algorithm to decide whether a graph G is full outer k-planar with a better runtime dependence on the parameter k?*

Last but not least:

▶ **Question 22.** *What is the complexity of outer k-quasi-planarity testing?*

In general k-quasi-planarity testing has notoriously eluded the efforts to establish its computational complexity even for $k = 3$ (or simply, quasi-planarity testing, with respect to our definition). Recently, Angelini et al. [10] showed that testing 2-level quasi-planarity (i.e., quasi-planarity for a bipartite graph in a 2-layer layout, where the vertices of each partition are on one of two parallel lines and the edges are drawn between the lines) is NP-complete, making first progress in this direction. Can this be adapted to show NP-completeness for testing outer k-quasi-planarity for any $k \geq 3$?

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A SAT formulations

In the following two sections we describe SAT formulations that can be used to test whether a given graph is outer quasi-planar (Section A.1) and to compute its page number (Section A.2). We present the formulas in first-order logic. After transformation to Boolean logic, the resulting formulas can be solved using, e.g., MiniSat (see `http://minisat.se/`).

A.1 Outer quasi-planarity checker

In this section, we describe a logical formula for testing whether a given graph G is outer quasi-planar (that is, outer 3-quasi-planar). We used this formula in order to prove Propositions 10(d) and 11 in Section 3. An outer quasi-planar embedding corresponds to a circular order of the vertices. If we cut a circular order at some vertex to turn the circular into a linear order, the edge crossing pattern remains the same. Therefore, we look for a linear order. For any pair (u, v) of vertices of G with $u \neq v$, we introduce a Boolean variable $x_{u,v}$ that expresses whether vertex u is before v in the linear order. In addition, for any pair (e, e') of edges of G with $e \neq e'$, we introduce a Boolean variable $y_{e,e'}$ that expresses whether edge e crosses edge e'. Now we list the clauses of our SAT formula.

$$x_{u,v} \wedge x_{v,w} \Rightarrow x_{u,w} \qquad \text{for each } u \neq v \neq w \neq u \in V(G); \tag{2}$$

$$x_{u,v} \Leftrightarrow \neg x_{v,u} \qquad \text{for each } u \neq v \in V(G); \tag{3}$$

$$x_{u,u'} \wedge x_{u',v} \wedge x_{v,v'} \Rightarrow y_{e,e'} \qquad \text{for each } e = (u, v) \neq e' = (u', v') \in E(G); \tag{4}$$

$$\neg(y_{e_1,e_2} \wedge y_{e_1,e_3} \wedge y_{e_2,e_3}) \qquad \text{for each } e_1, e_2, e_3 \in E(G) \text{ with different endpoints.} \tag{5}$$

The first two sets of clauses describe the linear order: transitivity (2) and anti-symmetry (3). Clauses (4) realize the intended meaning of the variables $y_{e,e'}$. Finally, clauses (5) ensure that no three edges pairwise cross.

A.2 Page number checker

In this section we provide a SAT formula that, given a graph G and an integer $k > 0$, has a satisfying truth assignment if and only if G has page number at most k. A similar SAT formulation has been implemented by Pupyrev [54] (see `http://be.cs.arizona.edu/`). For completeness, we list the constraints that we used in order to compute the page number of G_{ageev} (see Question 19). We find a linear order of the vertices that corresponds to a k-page embedding. For every pair (u, v) of vertices of G with $u \neq v$, we introduce a Boolean variable $x_{u,v}$ (as in Section A.1) that expresses whether u is before v in the linear order. For every edge e of G and every page $i \in \mathcal{P} = \{1, \ldots, k\}$, we introduce a Boolean variable $p_{i,e}$ that expresses whether edge e is on page i. Now we list the clauses of our SAT formula.

$$(x_{u,v} \wedge x_{v,w}) \Rightarrow x_{u,w} \qquad \text{for each } u \neq v \neq w \neq u \in V(G); \tag{6}$$

$$x_{u,v} \Leftrightarrow \neg x_{v,u} \qquad \text{for each } u \neq v \in V(G); \tag{7}$$

$$\bigvee_{i \in \mathcal{P}} p_{i,e} \qquad \text{for each } e \in E(G); \tag{8}$$

$$(p_{i,e} \wedge p_{i,e'}) \Rightarrow \neg(x_{u,u'} \wedge x_{u',v} \wedge x_{v,v'}) \qquad \text{for each } i \in \mathcal{P}, u \neq u', v \neq v' \in V(G), \tag{9}$$

$$e = (u, v), e' = (u', v') \in E(G) \tag{10}$$

The first two sets of clauses, (6)–(7), are the same as clauses (2)–(3) since they describe the linear order. Clauses (8) guarantee that every edge is on some page. Clauses (9) ensure that two edges that have different endpoints and lie on the same page do not cross. (If two edges share an endpoint, they cannot cross.)