The Tripartite-Circle Crossing Number of Graphs With Two Small Partition Classes

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Abstract

A tripartite-circle drawing of a tripartite graph is a drawing in the plane, where each part of a vertex partition is placed on one of three disjoint circles, and the edges do not cross the circles. The tripartite-circle crossing number of a tripartite graph is the minimum number of edge crossings among all its tripartite-circle drawings. We determine the exact value of the tripartite-circle crossing number of $K_{a,b,n}$, where $a, b \leq 2$.

Keywords and phrases graph drawing, crossing number, circle drawing, 3-circle drawing, complete tripartite graph

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1 Introduction

The *crossing number* of a graph G, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings over all drawings of G in the plane. There are multiple variants to the problem of finding the crossing number of a graph. Some problems consider drawings on other surfaces or in other spaces, like drawings on the torus, a cylinder, or a k-page book. Others restrict the minimum to specific types of drawings, like rectilinear or geometric drawings, where the edges must be straight line segments. Drawings with few crossings have been studied in connection with readability and VLSI chip design [17]. See [21] for a survey of crossing number variants and some of their applications.

Exact crossing numbers are unknown even for very special graph classes. Famous examples of these are the long-standing Harary-Hill Conjecture on the crossing number of the complete graph K_n [11, 10] and the Zarankiewicz Conjecture on the complete bipartite graph $K_{m,n}$ [24]. Among the most studied drawings of complete graphs are 2-page book drawings (or 1-*circle drawings*) and cylindrical drawings (or 2-*circle drawings*). This prompted the study of *k-circle drawings* of graphs [7], that is, drawings in the plane where the vertices are placed on k specified circles and the edges cannot cross these circles. Of particular interest are *k-partite-circle drawings*, where we further require that the vertices on each circle form an independent set. The minimum number of crossings among all k-partite-circle drawings of a graph G is known as the *k-partite-circle crossing number* of G and is denoted by $\mathrm{cr}_{ⓚ}(G)$.

This crossing number has been studied for complete bipartite and tripartite graphs. In 1997, Richter and Thomassen [20] settled the balanced case for complete bipartite graphs by showing that $\mathrm{cr}_{②}(K_{n,n}) = n\binom{n}{3}$. Ábrego, Fernández-Merchant, and Sparks [1] generalized this result to all complete bipartite graphs. The exact expression for this crossing number was complicated as it was given in terms of summations involving floor and ceiling functions. It was recently simplified to

$$\mathrm{cr}_{②}(K_{m,n}) = \binom{n}{2}\binom{m}{2} - 1/12 \cdot (n^2m^2 - n^2 - m^2 + \gcd(m,n)^2) \tag{1}$$

by Ábrego and Fernández-Merchant in [2]. In previous work [4], we proved lower and upper bounds on $\mathrm{cr}_{③}(K_{m,n,p})$; the implied lower and upper bounds for $K_{n,n,n}$ are of orders $5/4 \cdot n^4$ and $6/4 \cdot n^4$, respectively.

In this paper, we study $\mathrm{cr}_{③}(K_{m,n,p})$ for small values of m and n. First, for $m = n = 1$, we show that crossing-minimal tripartite-circle drawings are in one-to-one correspondence with crossing-minimal bipartite-circle drawings of $K_{2,n}$.

▶ **Proposition 1.** *For every positive integer n,*

$$\mathrm{cr}_{③}(K_{1,1,n}) = \mathrm{cr}_{②}(K_{2,n}) = \lceil 1/4 \cdot n(n-2) \rceil.$$

The next smallest case, $K_{1,2,n}$, behaves differently. We show that $\mathrm{cr}_{③}(K_{1,2,n}) \approx 3/4 \cdot n^2$, which is smaller than $\mathrm{cr}_{②}(K_{3,n}) \approx 5/6 \cdot n^2$.

▶ **Theorem 2.** *For every integer $n \geq 2$,*

$$\mathrm{cr}_{③}(K_{1,2,n}) = \lfloor 3/4 \cdot n^2 \rfloor - n.$$

In previous work, we established that $\mathrm{cr}_{③}(K_{2,2,2}) = 3$; for an illustration we refer to Figure 12(c) in [4]. As our main result, we establish $\mathrm{cr}_{③}(K_{2,2,n})$ for all $n \geq 3$.

▶ **Theorem 3.** *For every integer $n \geq 3$,*

$$\mathrm{cr}_{③}(K_{2,2,n}) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n - 3.$$

In comparison, the Zarankiewicz Conjecture on $\mathrm{cr}(K_{m,n})$ has been proved only when $\min(m,n) \leq 6$ by Kleitman in 1970 [15], and for $m = 7$ or $m = 8$ and $n \leq 10$ by Woodall in 1993 [22]. The current best lower bounds are by de Klerk et al. [6] and Norin and Zwols [19]. The crossing number of the complete tripartite graph is also unknown in general. In 2017, Gethner et al. [8] provided a drawing of $K_{m,n,p}$ with few crossings and conjectured that their drawing is crossing-optimal. The exact crossing numbers for small values of m and n are known only for $\mathrm{cr}(K_{1,n,p})$ with $n \le 5$ [12, 3, 14, 18] and for $\mathrm{cr}(K_{2,n,p})$ with $n \le 4$ [16, 3, 13]. For $\mathrm{cr}(K_{3,3,p})$, only asymptotically tight bounds are known [9]. A general conjecture on the crossing number of $K_{1,m,n}$ has been proven only under the hypothesis that the Zarankiewicz Conjecture holds [23]. See [5] for a survey.

A contrast between crossing numbers and circle crossing numbers can be seen in a comparison of the graphs $K_{2,2,n}$ and $K_{4,n}$, which differ by only four edges. While their crossing numbers are equal, as $\mathrm{cr}(K_{4,n}) = \mathrm{cr}(K_{2,2,n}) = 2\lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor = 1/2 \cdot n^2 + \Theta(n)$ [15, 16], their circle crossing numbers evaluate to $\mathrm{cr}_{③}(K_{2,2,n}) = 3/2 \cdot n^2 + \Theta(n)$ and $\mathrm{cr}_{②}(K_{4,n}) = 7/4 \cdot n^2 + \Theta(n)$ [1] and thus differ by a term quadratic in n.

Organization. The remainder of our paper is organized as follows: In Section 2, we introduce tools for counting the number of crossings based on previous work. In Section 3, we prove Proposition 1 regarding crossing-minimal tripartite-circle drawings of $K_{1,1,n}$ and prove Theorem 2 determining the tripartite-circle crossing number of $K_{1,2,n}$, and then describe all crossing-minimal drawings. In Section 4, we present our proof of Theorem 3, yielding the tripartite-circle crossing number of $K_{2,2,n}$. We conclude in Section 5 with a discussion and open problems.

2 Tools for counting the number of crossings

Our tools for counting the number of crossings build on our previous work in [4]. For a self-contained presentation, we give a concise summary.

Drawings that minimize the number of crossings are known to be *simple drawings*, that is, drawings where edges are simple curves and any two edges have at most one point in common, which is either a vertex or a crossing. In a tripartite-circle drawing of $K_{m,n,p}$, we label the three circles M, N, and P, and their numbers of vertices are m, n, and p, respectively. For an illustration consider Figure 1.

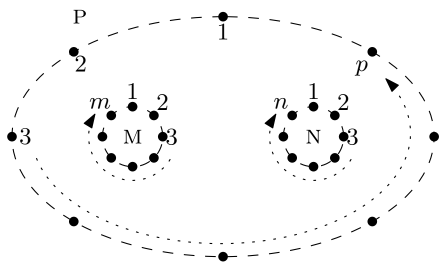
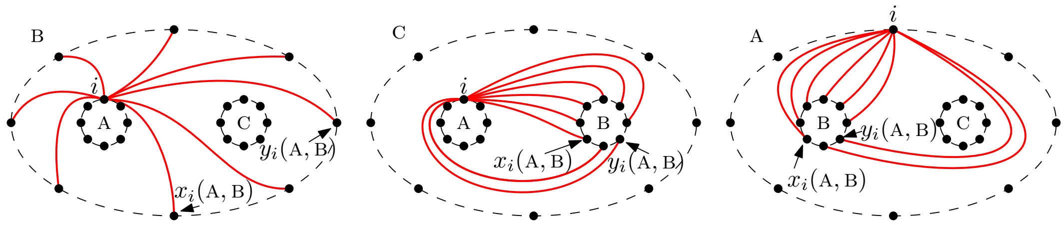


Figure 1 The vertices on the circles M and N are labeled clockwise; the vertices of the circle P are labeled counterclockwise.

By using projective transformations and the fact that in a tripartite-circle drawing of a complete tripartite graph the three underlying circles cannot be pairwise nested (they are not to be crossed by any edges), we may consider only the drawings where the *outer* circle P contains the *inner* circles M and N, see Figure 1. In such a drawing, we label the vertices on circles M and N in clockwise order and the vertices on circle P in counterclockwise order. Likewise, we read arcs of circles in clockwise order for inner circles and in counterclockwise order for outer circles.

2.1 Defining the x- and y-labels

As progressively defined in [20], [1], and [4], we use the following labels. Let A, B, and C be the three circles and i be a vertex on A. The star formed by all edges from i to B together with circle B partitions the plane into disjoint regions, shown in Figure 2. Exactly one of these regions contains circle A. This region is enclosed by two edges from i to B and an arc on B between two consecutive vertices. We define $x_i(\text{A},\text{B})$ as the second of these vertices (in clockwise or counterclockwise order depending on whether B is an inner or outer circle, respectively). Similarly, there is exactly one region defined by the star from i that contains the third circle C, and the second vertex along circle B (clockwise or counterclockwise as before) on the boundary of this region is $y_i(\text{A},\text{B})$. If the two circles are clear from the context, we may also write x_i or y_i.



(a) Nested: A is inside B (b) Non-Nested: A is beside B (c) Nested: B is inside A

Figure 2 The definitions of $x_i(\text{A},\text{B})$ and $y_i(\text{A},\text{B})$ illustrated for three cases.

2.2 Counting the crossings

We introduce notation necessary to state Theorem 7 in [4] that provides a precise formula for the number of crossings in a tripartite-circle drawing of $K_{m,n,p}$. For vertices k and ℓ on a circle with n vertices numbered $1,\dots,n$ clockwise (respectively, counterclockwise), let

$$d_n(k,\ell) := \ell - k \mod n$$

denote the *distance* from k to ℓ in clockwise (respectively, counterclockwise) order on the circle. For any $u,v \in \{1,2,\dots,n\}$, define

$$f_n(u,v) := \binom{d_n(u,v)}{2} + \binom{n - d_n(u,v)}{2}.$$

Throughout the paper, we use the facts that for all u and v we have $f_n(u,v) = f_n(v,u)$ and

$$f_n(u,v) \geq \binom{\lfloor \frac{n}{2} \rfloor}{2} + \binom{\lceil \frac{n}{2} \rceil}{2} = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor. \qquad (2)$$

As in [4], for vertices i and j on an inner (respectively, outer) circle A, we write $[i,j]$ for the arc of A read clockwise (respectively, counterclockwise) from i to j. We define $[i,j)$, $(i,j]$, and (i,j) similarly, where a square bracket denotes inclusion of the endpoint, and a parenthesis denotes exclusion of the endpoint.

A *cyclic assignment* of (A,B,C) to (M,N,P) is one triple of the set $\mathfrak{t} := \{$(M,N,P), (N,P,M), (P,M,N)$\}$. Let the numbers of vertices on the circles A, B, and C be denoted by a, b, and c, respectively. The following theorem counts the total number of crossings.

▶ **Theorem 4** (Theorem 7, [4]). *The number of crossings in a simple tripartite-circle drawing of $K_{m,n,p}$ is given by*

$$\sum_{(A,B,C)\in \mathfrak{t}} \left(\sum_{\substack{i<j\\ i,j\in A}} f_b\big(x_i(A,B), x_j(A,B)\big) + \sum_{\substack{i\in A\\ j\in B}} f_c\big(y_i(A,C), y_j(B,C)\big) \right).$$

In this result (see Lemmas 5 and 6 in [4]), $\sum_{\substack{i<j\\ i,j\in \mathrm{A}}} f_b\big(x_i(\mathrm{A,B}), x_j(\mathrm{A,B})\big)$ counts the number of crossings determined by edges incident to one vertex on circle A and another on circle B; whereas $\sum_{\substack{i\in \mathrm{A}\\ j\in \mathrm{B}}} f_c\big(y_i(\mathrm{A,C}), y_j(\mathrm{B,C})\big)$ counts the number of crossings determined by an edge between circles A and C with an edge between circles B and C.

3 Proofs of Proposition 1 and Theorem 2

In this section, we determine the exact values of $\mathrm{cr}_{③}(K_{1,1,n})$ and $\mathrm{cr}_{③}(K_{1,2,n})$.

▶ **Proposition 1.** *For every positive integer n,*

$$\mathrm{cr}_{③}(K_{1,1,n}) = \mathrm{cr}_{②}(K_{2,n}) = \lceil 1/4 \cdot n(n-2) \rceil.$$

Proof. Let D be any crossing-minimal drawing of $K_{1,1,n}$. By minimality, D is simple; in particular, any two edges share at most one point. Let v and w be the vertices in the 1-vertex parts of $K_{1,1,n}$, and let $e = \{v, w\}$ denote their edge. Because any other edge e' of $K_{1,1,n}$ is incident to either v or w, e' does not cross e in D; otherwise e and e' would share two points. Consequently, e is not involved in any crossing and we can easily replace e with two parallel curves along e from v to w (representing a circle). This yields a bipartite-circle drawing of $K_{2,n}$ with the same number of crossings. By the reverse substitution, any bipartite-circle drawing of $K_{2,n}$ yields a tripartite-circle drawing of $K_{1,1,n}$ with the same number of crossings. Thus the two crossing numbers are equal. Finally, $\mathrm{cr}_{②}(K_{2,n}) = \binom{n}{2} - 1/12(4n^2 - 4 - n^2 + \gcd(2,n)^2) = \lceil 1/4 \cdot n(n-2)\rceil$ by Equation (1). ◀

It turns out that $\mathrm{cr}_{③}(K_{1,2,n}) \approx 3/4 \cdot n^2$, which is smaller than $\mathrm{cr}_{②}(K_{3,n}) \approx 5/6 \cdot n^2$.

▶ **Theorem 2.** *For every integer $n \geq 2$,*

$$\mathrm{cr}_{③}(K_{1,2,n}) = \lfloor 3/4 \cdot n^2 \rfloor - n.$$

The proof of Theorem 2 is a direct consequence of the following three lemmas: Lemma 5 takes care of the cases $2 \leq n \leq 4$, and Lemmas 6 and 7 settle the upper and lower bounds for $n \geq 5$, respectively.

▶ **Lemma 5.** *It holds that $\mathrm{cr}_{③}(K_{1,2,2}) = 1, \mathrm{cr}_{③}(K_{1,2,3}) = 3$, and $\mathrm{cr}_{③}(K_{1,2,4}) = 8$.*

Proof. For $n = 2, 3, 4$, Figure 3 presents tripartite-circle drawings of $K_{1,2,n}$ with 1, 3, and 8 crossings, respectively, which prove the upper bounds.

For the lower bounds, we use the known crossing numbers $\mathrm{cr}(K_5) = 1$, $\mathrm{cr}(K_6) = 3$, and $\mathrm{cr}(K_7) = 9$. For any fixed drawing D, we denote its number of crossings by $\mathrm{cr}(D)$. For $n = 2, 3, 4$, let D' be a crossing-minimal tripartite-circle drawing of $K_{1,2,n}$. We construct a drawing D of K_{n+3} as an extension of D'. For $n = 2, 3$, it suffices to add edges along the arcs of the n-vertex circle and the 2-vertex circle as depicted by the gray edges in Figures 3(a) and 3(b), introducing no new crossings. Thus $\mathrm{cr}(K_{n+3}) \leq \mathrm{cr}(D) = \mathrm{cr}(D') = \mathrm{cr}_{③}(K_{1,2,n})$, so

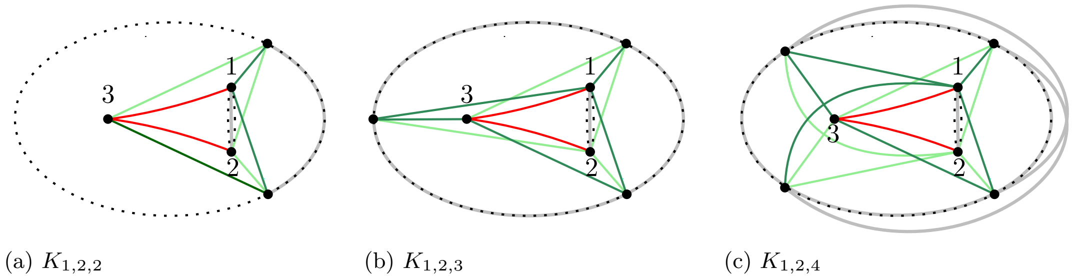


Figure 3 Crossing-minimal tripartite-circle drawings of $K_{1,2,n}$ for $n = 2, 3, 4$; crossing-optimal drawings of the complete graph K_{n+3} are obtained by adding gray edges.

$\text{cr}_{③}(K_{1,2,2}) = 1$ and $\text{cr}_{③}(K_{1,2,3}) = 3$. For $n = 4$, we obtain a drawing D of K_7 extending D', shown in Figure 3(c), by adding one edge along a semicircle of the 2-vertex circle, four edges along the arcs of the 4-vertex circle, and two additional edges which cross each other but no other edges. Then $9 = \text{cr}(K_7) \le \text{cr}(D) = \text{cr}(D')+1 = \text{cr}_{③}(K_{1,2,4})+1$, so $\text{cr}_{③}(K_{1,2,4}) = 8$. ◀

For the rest of this section, we label the point on the 1-vertex circle M by 3, and the points of the 2-vertex circle N by 1 and 2, in such a way that the interior of triangle $\{1, 2, 3\}$ (read clockwise) does not contain the n-vertex circle P (see Figures 4 and 5).

▶ **Lemma 6.** *For every integer $n \ge 5$, $\text{cr}_{③}(K_{1,2,n}) \le \lfloor 3/4 \cdot n^2 \rfloor - n$.*

Proof. The family of straight-line drawings in Figure 4 yields the upper bound having exactly $\lfloor 3/4 \cdot n^2 \rfloor - n$ crossings. In the drawing, we group all vertices on circle P except w into four sets A, B, C, and D whose sizes depend on the parameter t. Let $2 \le t \le \lceil (n-1)/2 \rceil$, and $|A| = t - 1$, $|B| = \lfloor \frac{n+1}{2} \rfloor - t$, $|C| = t$, and $|D| = n - \lfloor \frac{n+1}{2} \rfloor - t$; see Figure 4. (Although a single drawing would be enough to settle the bound, this family of drawings will be part of the full classification of crossing-optimal drawings in Section 3.1.)

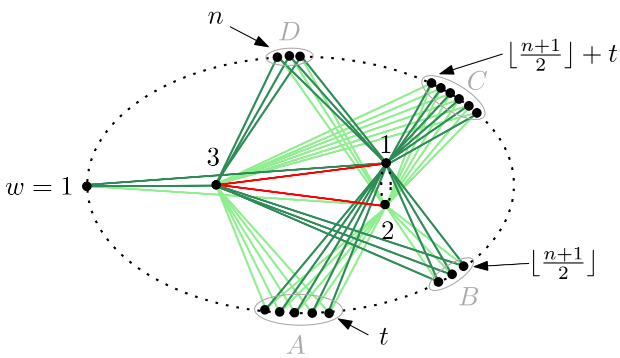


Figure 4 A crossing-minimal tripartite-circle drawing of $K_{1,2,n}$ for all $n \ge 4$ and $2 \le t \le \lceil (n-1)/2 \rceil$.

We color the edge between vertices 1 and 3 and the edge between vertices 2 and 3 red; all other edges are green. There are $|A \cup D|$ crossings with red edges. The green-green crossings are determined by two vertices in $\{1, 2, 3\}$ and two vertices on P: more precisely, by 1, 2, and any two vertices in $A \cup D \cup \{w\}$ or in $B \cup C$; by 1, 3, and any two vertices in $C \cup D \cup \{w\}$ or in $A \cup B$; or by 2, 3 and any two vertices in $A \cup B \cup \{w\}$ or in $C \cup D$. Since $|A \cup D \cup \{w\}| = |C \cup D| = \lceil (n + 1)/2 \rceil - 1$ and $|A \cup B \cup \{w\}| = |B \cup C| = \lfloor (n + 1)/2 \rfloor$, the total number of crossings is

$$\left\lceil\frac{n+1}{2}\right\rceil-2+2\binom{\lceil\frac{n+1}{2}\rceil-1}{2}+2\binom{\lfloor\frac{n+1}{2}\rfloor}{2}+\binom{\lfloor\frac{n+1}{2}\rfloor-1}{2}+\binom{\lceil\frac{n+1}{2}\rceil}{2}=\lfloor 3/4\cdot n^2\rfloor-n. \blacktriangleleft$$

▶ **Lemma 7.** *For every integer $n \geq 5$, $\mathrm{cr}_{③}(K_{1,2,n}) \geq \lfloor 3/4 \cdot n^2 \rfloor - n$.*

Proof. Start with an arbitrary simple tripartite-circle drawing of $K_{1,2,n}$; for an example consider Figure 5. For $i = 1, 2$, let U_i be the set of vertices u on circle P such that iu crosses $j3$, where $\{i, j\} = \{1, 2\}$. Note that $U_1 = [y_1, x_1)$ and $U_2 = [x_2, y_2)$ and so the number of crossings with the edges 13 and 23 is $d_n(y_1, x_1) + d_n(x_2, y_2)$. By Theorem 4, there are $f_n(x_1, x_2)$ crossings of edges $1u$ and $2v$, $f_n(y_1, y_3)$ crossings of edges $1u$ and $3v$, and $f_n(y_2, y_3)$ crossings of edges $2u$ and $3v$, where u and v are vertices on circle P.

Therefore, the total number of crossings is

$$d_n(y_1, x_1) + d_n(x_2, y_2) + f_n(x_1, x_2) + f_n(y_1, y_3) + f_n(y_2, y_3). \tag{3}$$

In order to simplify the presentation, we write $d_x := d_n(x_1, x_2)$ and $d_y := d_n(y_2, y_1)$.

We show that for fixed y_1 and y_2, the value of $f_n(y_1, y_3) + f_n(y_2, y_3)$ is minimized when y_3 divides the longer of the intervals $[y_2, y_1)$ and $[y_1, y_2)$ in half. For $\{i, j\} = \{1, 2\}$ and $y_3 \in (y_i, y_j]$, we write $D := d_n(y_3, y_j)$ and $c_i := d_n(y_j, y_i)$. Using the identity $\binom{x}{2} + \binom{n-x}{2} = \binom{n}{2} - x(n - x)$, we obtain

$$\begin{aligned} f_n(y_1, y_3) + f_n(y_2, y_3) &= 2\binom{n}{2} - (n - D)D - (n - c_i - D)(c_i + D) \\ &= 2D^2 + 2(c_i - n)D + c_i(c_i - n) + 2\binom{n}{2}. \end{aligned}$$

This is a quadratic function in D and thus minimized when $D \in \{\lfloor \frac{n-c_i}{2} \rfloor, \lceil \frac{n-c_i}{2} \rceil\}$, so

$$f_n(y_1, y_3) + f_n(y_2, y_3) \geq 2\binom{n}{2} - \left\lfloor \frac{n^2 - \min(d_y, n - d_y)^2}{2} \right\rfloor. \tag{4}$$

Since the drawing is simple, and the edges $1u$ and $2v$ cross for any $u \in U_1$ and $v \in U_2$ (see Figure 5), the sets U_1 and U_2 must be disjoint. If $U_i = \emptyset$, then $x_i = y_i$. If $U_1 \neq \emptyset$, then its first and last points counterclockwise along the circle P are y_1 and $x_1 - 1$, respectively. Similarly, if $U_2 \neq \emptyset$, then its first and last points counterclockwise along the circle P are x_2 and $y_2 - 1$, respectively. We consider three cases.

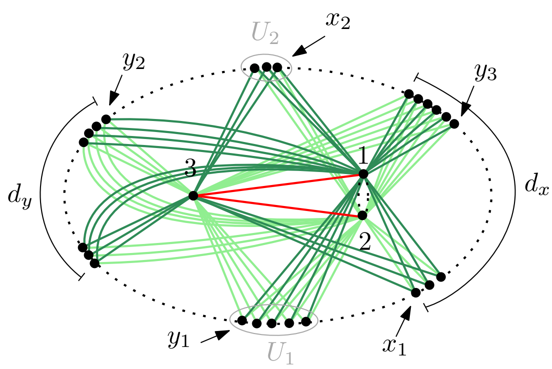


Figure 5 A simple tripartite-circle drawing of $K_{1,2,n}$ to illustrate Lemma 7.

▶ Case 7.1. U_1 and U_2 are both empty. Then $x_1 = y_1$ and $x_2 = y_2$, which implies $d_n(y_1, x_1) + d_n(x_2, y_2) = 0$ and $d_x + d_y = n$. Let $d = \min\{d_x, d_y\}$. Using Equation (4), Expression (3) becomes

$$f_n(x_1,x_2) + f_n(y_1,y_3) + f_n(y_2,y_3) \geq 3\binom{n}{2} - d(n-d) - \left\lfloor \frac{n^2-d^2}{2} \right\rfloor$$
$$= 3\binom{n}{2} - \left\lfloor \frac{(n-d)(n+3d)}{2} \right\rfloor \geq 3\binom{n}{2} - \lfloor 2/3 \cdot n^2 \rfloor \geq \lfloor 3/4 \cdot n^2 \rfloor - n,$$

where the last inequality is strict for $n = 5$ and $n \geq 7$. Therefore, this case yields a crossing-minimal drawing only when $n = 6$.

It remains to consider the case when at least one of the sets U_1 or U_2 is nonempty. By symmetry, we assume that U_1 is nonempty and analyze two cases.

▶ Case 7.2. U_1 is nonempty and $x_2 \in [x_1, y_1]$. If U_2 is nonempty, since U_1 and U_2 are disjoint, the vertices y_1, x_1, x_2, y_2 appear in this order counterclockwise along the n-vertex circle (some of them could overlap). If $U_2 = \emptyset$, then $x_2 = y_2$ and so $y_1, x_1, x_2 = y_2$ appear in this order counterclockwise along the n-vertex circle. In both cases,

$$d_n(y_1,x_1) + d_x + d_n(x_2,y_2) + d_y = n.$$

Since $U_1 \neq \emptyset$, then $d_n(y_1, x_1) \geq 1$ and so $d_x + d_y \leq n - 1$. Hence, Expression (3) becomes

$$n - d_x - d_y + f_n(x_1,x_2) + f_n(y_1,y_3) + f_n(y_2,y_3)$$
$$\geq n - d_x - d_y + 3\binom{n}{2} - d_x(n-d_x) - \left\lfloor \frac{n^2 - \min(d_y, n-d_y)^2}{2} \right\rfloor$$
$$= 3\binom{n}{2} - d_x\,(n+1-d_x) - \left\lfloor \begin{cases} (n-d_y)(n-2+d_y)/2 & \text{if } d_y \leq n/2 \\ (n^2-(n-d_y)(n-d_y+2))/2 & \text{if } d_y > n/2 \end{cases} \right\rfloor . \tag{5}$$

Assume first that $d_y \leq n/2$. For $n \geq 4$, Expression (5) is minimized when $d_x = \lfloor \frac{n+1}{2} \rfloor$ and $d_y = 1$ implying

$$n - d_x - d_y + f_n(x_1,x_2) + f_n(y_1,y_3) + f_n(y_2,y_3)$$
$$\geq 3\tbinom{n}{2} - \lfloor \tfrac{n+1}{2} \rfloor \lceil \tfrac{n+1}{2} \rceil - \lfloor 1/2 \cdot (n-1)^2 \rfloor \geq \lfloor 3/4 \cdot n^2 \rfloor - n.$$

Assume now that $d_y > n/2$. Then

$$n - d_x - d_y + f_n(x_1,x_2) + f_n(y_1,y_3) + f_n(y_2,y_3)$$
$$\geq 3\tbinom{n}{2} - d_x\,(n+1-d_x) - 1/2 \cdot (n^2 - (n-d_y)(n-d_y+2)). \tag{6}$$

Recall that $d_x + d_y \leq n - 1$ which implies that $n - d_y \geq d_x + 1$. This replacement allows Expression (6) to be minimized with respect to d_x, and occurs when $d_x = \frac{n-1}{3}$. Now we have that $d_y \leq \frac{2(n-1)}{3}$. Therefore, Expression (6) is minimized when $d_x = \frac{n-1}{3}$ and $d_y = \frac{2(n-1)}{3}$, and the inequality becomes

$$n - d_x - d_y + f_n(x_1,x_2) + f_n(y_1,y_3) + f_n(y_2,y_3)$$
$$\geq 3\tbinom{n}{2} - \tfrac{n-1}{3}\left(n+1-\tfrac{n-1}{3}\right) - \tfrac{1}{2}\left(n^2 - \left(n - \tfrac{2(n-1)}{3}\right)\left(n - \tfrac{2(n-1)}{3} + 2\right)\right)$$
$$= 1/6 \cdot (5n^2 - 7n + 8) > \lfloor 3/4 \cdot n^2 \rfloor - n.$$

Note that this minimization may be at values when d_x and d_y are not integers. Alternate pairs of values $d_x = \frac{n-3}{3}$ and $d_y = \frac{2n}{3}$ or $d_x = \frac{n-2}{3}$ and $d_y = \frac{2n-1}{3}$ depending on divisibility of $n \pmod 3$ also yield the desired strict inequality.

▶ Case 7.3. U_1 is nonempty and $x_2 \in (y_1, x_1)$. In this case, $U_2 = \emptyset$ because U_1 and U_2 are disjoint. In this case $x_2 = y_2$ and so $d_y \geq 1$. Then

$$d_n(y_1, x_1) + d_n(y_2, x_2) = d_n(y_1, x_1) = n - d_x + d_y.$$

If $d_y > n/2$, then $d_n(y_1, x_1) + d_n(y_2, x_2) \geq d_y > n/2$. Using Equation (2), we have

$$\begin{aligned} & d_n(y_1, x_1) + d_n(x_2, y_2) + f_n(x_1, x_2) + f_n(y_1, y_3) + f_n(y_2, y_3) \\ & > \left\lfloor \frac{n}{2} \right\rfloor + 3 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \geq \lfloor 3/4 \cdot n^2 \rfloor - n. \end{aligned}$$

If $1 \leq d_y \leq n/2$, then $\min(d_y, n - d_y) = d_y$ and so Expression (3) can be written as

$$\begin{aligned} & n + d_y - d_x + f_n(x_1, x_2) + f_n(y_1, y_3) + f_n(y_2, y_3) \\ & \geq n + d_y - d_x + 3\binom{n}{2} - d_x(n - d_x) - \left\lfloor \frac{n^2 - d_y^2}{2} \right\rfloor \\ & = 3\binom{n}{2} - d_x\,(n + 1 - d_x) - \left\lfloor \frac{(n + d_y)(n - 2 - d_y)}{2} \right\rfloor. \end{aligned} \tag{7}$$

For $n \geq 3$, Expression (7) is minimized when $d_x = \lfloor \frac{n+1}{2} \rfloor$ or $\lceil \frac{n+1}{2} \rceil$ and $d_y = 1$. Thus

$$\begin{aligned} & n - d_x + d_y + f_n(x_1, x_2) + f_n(y_1, y_3) + f_n(y_2, y_3) \\ & \geq 3\tbinom{n}{2} - \lfloor \tfrac{n+1}{2} \rfloor \lceil \tfrac{n+1}{2} \rceil - \lfloor 1/2 \cdot (n+1)(n-3) \rfloor > \lfloor 3/4 \cdot n^2 \rfloor - n. \end{aligned}$$ ◀

3.1 List of all crossing-minimal tripartite circle drawings of $K_{1,2,n}$

We use the proof of the lower bound of Theorem 2 (Lemma 7) to track all cases when equality holds. In Case 7.1, the inequality becomes an identity for $n = 2, 3$, and 4. The other two inequalities are tight when $\min\{d_x, d_y\} = \lfloor n/3 \rfloor$ or $\lceil n/3 \rceil$. Figures 6(a)-(e) show the corresponding crossing-minimal drawings of $K_{1,2,n}$.

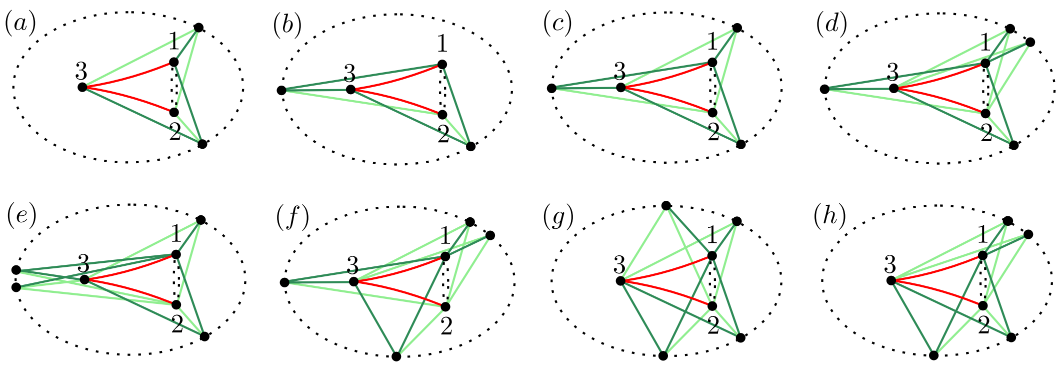


Figure 6 All crossing-minimal tripartite-circle drawings of $K_{1,2,n}$ for $n = 2, 3$, and 4.

When $n = 6$, the inequality of Case 7.1 is strict unless $d = 2$ which occurs if $(d_x, d_y) = (4, 2)$. This drawing is given in Figure 7.

The other three drawings of $K_{1,2,4}$, depicted in Figures 6(f)-(h), come from Case 7.2. In this case, Inequality (6) is still strict for $n = 2, 3, 4$. Equation (5) for $d_y \leq n/2$ is tight when y_3 divides the longest of the intervals $[y_2, y_1]$ or $[y_1, y_2]$ in almost half (so that Equation (4)

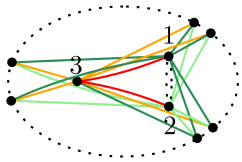


Figure 7 Crossing-minimal tripartite-circle drawing of $K_{1,2,6}$.

is also tight) and

$$(d_x, d_y) = \begin{cases} (\lfloor \frac{n+1}{2} \rfloor, 1), (\lceil \frac{n+1}{2} \rceil, 1) & \text{if } n \geq 5, \\ (\frac{n}{2}, 0) & \text{if } n \geq 2 \text{ even}, \\ (\frac{n}{2}, 2), (\frac{n}{2}+1, 2) & \text{if } n \geq 8 \text{ even}, \\ (3,2), (4,0) & \text{if } n = 6, \\ (2,1), (3,0) & \text{if } n = 4. \end{cases}$$

Crossing-optimal configurations exist for each of these situations. Figure 4 (with $A = U_1, B = V_1, C = V_2, D = U_2$) shows the case $(d_x, d_y) = (\lfloor (n+1)/2 \rfloor, 1)$ for $n \geq 4$ (and y_3 slightly closer to y_1 than y_2 when n is even). Figure 8 shows the case $(d_x, d_y) = (n/2, 0)$ for $n \geq 2$ even; note that these two figures in the particular case when $n = 4$ are shown in Figures 6(f) and 6(g). Figure 6(h) corresponds to the case $(d_x, d_y) = (3, 0)$ and $n = 4$.

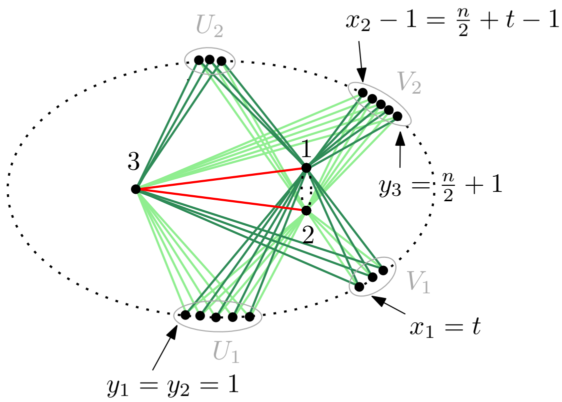


Figure 8 A crossing-minimal tripartite-circle drawing of $K_{1,2,n}$ for even $n \geq 4$ and $2 \leq t \leq n/2+1$.

Finally, Case 7.3 requires $n \geq 3$, and the last inequality is still strict for $n = 3$ and 4. Hence, there are no crossing-minimal drawings in this case.

4 Proof of Theorem 3

In this section, we determine the tripartite-circle crossing number of $K_{2,2,n}$.

▶ **Theorem 3.** *For every integer* $n \geq 3$,

$$\text{cr}_{③}(K_{2,2,n}) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n - 3.$$

The proof of Theorem 3 is a direct consequence of the results in Sections 4.1 and 4.2. First, in Section 4.1, we give a construction to show the upper bound. Then, in Section 4.2, we show the lower bound by analyzing several cases to show that our construction actually minimizes the number of crossings of $K_{2,2,n}$.

4.1 Upper bound of Theorem 3

The following lemma states the upper bound of Theorem 3.

▶ **Lemma 8.** *For every integer $n \geq 3$,*

$$\mathrm{cr}_{③}(K_{2,2,n}) \leq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n - 3.$$

Proof. In Figure 9, we define two drawings that achieve the bound for odd values of $n \geq 3$ in drawing (a) and for all values of $n \geq 4$ in drawing (b). Both drawings have M and N as the inner circles with two vertices each. We color the edges between M and N red and all others green. We divide the vertices on P into groups which are called A, B, C, D. In drawing (a), group A has $\lfloor \frac{n}{2} \rfloor$ vertices, consisting of those labeled 1 through $\lfloor \frac{n}{2} \rfloor$ and group B has the remaining vertices, $\lceil \frac{n}{2} \rceil$ of them; groups C and D are empty. In drawing (b), each group consists of consecutive vertices moving counter clockwise; group A has $\lfloor \frac{n}{4} \rfloor$ vertices, beginning with vertex n, group B contains vertices $\lfloor \frac{n}{4} \rfloor$ through $\lfloor \frac{n}{2} \rfloor - 1$, group C contains vertices $\lfloor \frac{n}{2} \rfloor$ through $\lfloor \frac{3n}{4} \rfloor - 1$, and group D contains all remaining vertices, that is vertices $\lfloor \frac{3n}{4} \rfloor$ through $n-1$. All green edges are straight line segments except for $1x_1$ and $2x_2$ in (a) and $1y_2$, $2x_3$, $3y_4$, and $4x_1$ in (b). If these edges are replaced by straight line segments, the number of crossings is easy to determine. These replacements add two crossings to (a) and four to (b). For simplicity, we define $c_n := 6\left(\binom{\lfloor n/2 \rfloor}{2} + \binom{\lceil n/2 \rceil}{2}\right) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.

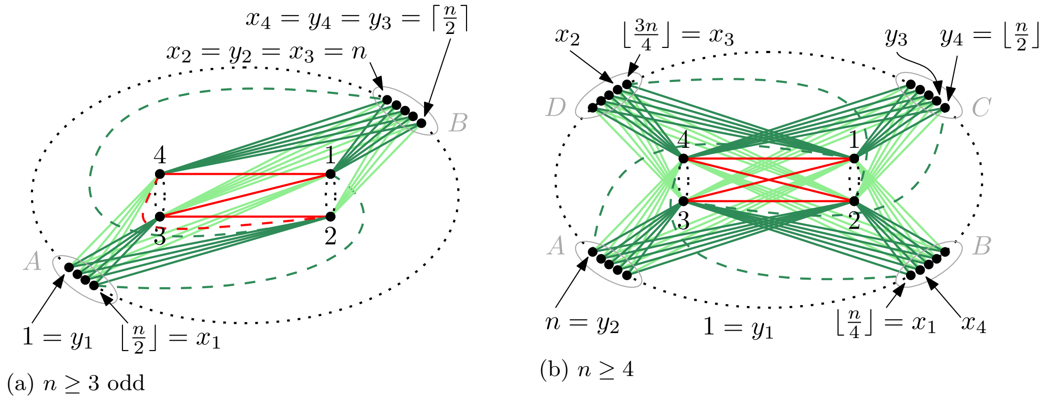


Figure 9 Two crossing-minimal tripartite-circle drawings of $K_{2,2,n}$, where all but at most four edges are straight line segments.

After replacing these (curved) green edges in Figure 9(a) by straight line segments, the drawing has $3\lfloor n/2 \rfloor + \lceil n/2 \rceil = 2n - 1$ red-green crossings (when n is odd) and no red-red crossing. In order to count the green-green crossings, note that every crossing is determined by any two vertices on the inner circles and any two vertices in A or any two vertices in B. Thus there are c_n green-green crossings and the number of crossings in the drawing displayed in Figure 9(a) is $c_n + 2n - 1 - 2 = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n - 3$.

Similarly, after the replacement in Figure 9(b), the drawing has $2n$ red-green crossings and 1 red-red crossing. The green-green crossings are determined by either two vertices on the same inner circle and any two vertices in $A \cup D$ or any two vertices on $B \cup C$; or by two vertices on different inner circles and any two vertices in $A \cup B$ or any two vertices on $C \cup D$. Note that $\{|A \cup D|, |B \cup C|\} = \{|A \cup B|, |C \cup D|\} = \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$ and thus there are c_n green-green crossings. The number of crossings in the drawing displayed in Figure 9(b) is $c_n + 2n + 1 - 4 = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n - 3$. ◀

4.2 Lower bound of Theorem 3

Now, we turn our attention towards proving the following lower bound, i.e., we want to show that the drawings given in Section 4.1 have the minimum number of crossings among tripartite-circle drawings.

▶ **Lemma 9.** *For every integer $n \geq 3$,*

$$\mathrm{cr}_{③}(K_{2,2,n}) \geq 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n - 3.$$

In order to prove Lemma 9, we need a series of lemmas. Lemma 10 categorizes all simple tripartite-circle drawings into Types 1, 2, 3, or 4. Lemma 11 provides expressions for the exact number of crossings in each of the four types; Lemmas 11, 12, and 13 give bounds for different components of the expressions in Lemma 11; Lemma 14 settles the lower bound for the drawings of Types 2, 3, 4 and Lemma 15 for the drawings of Type 1.

It is sufficient to consider simple tripartite-circle drawings. To analyze all such drawings, we partition the set of simple drawings of $K_{2,2,n}$ by the induced subdrawings of $K_{2,2,0}$ depicted in Figure 10(b).

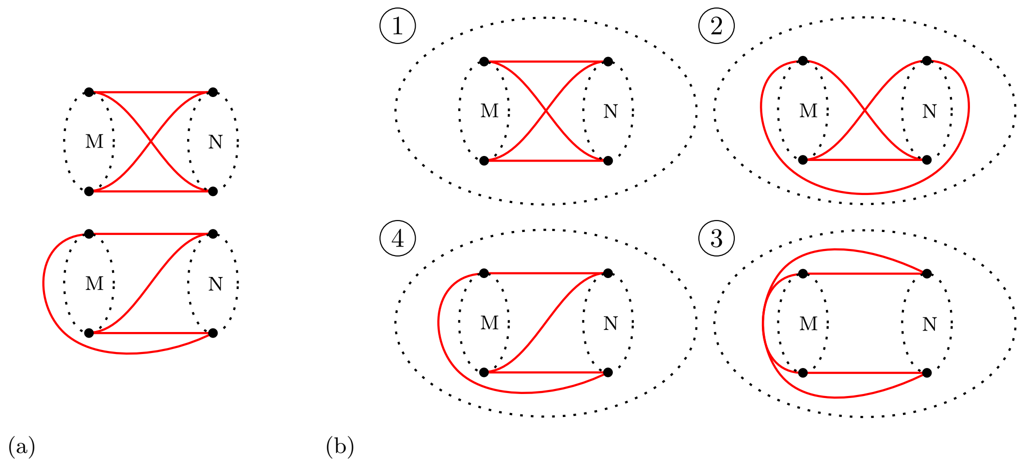


Figure 10 (a) The two simple bipartite-circle drawings of $K_{2,2}$. (b) The four simple tripartite-circle drawings of $K_{2,2,0}$.

▶ **Lemma 10.** *Up to topological equivalence, there are exactly four simple tripartite-circle drawings of $K_{2,2,0}$.*

Proof. Up to topological equivalence, there are two different simple bipartite-circle drawings of $K_{2,2}$ on the sphere, namely the ones depicted in Figure 10(a). The edges in the first drawing define five regions, and the edges in the second drawing define four regions.

Placing the third circle in each of these regions, then finding the equivalent drawing such that the third circle encloses M and N, yields four distinct drawings, shown in Figure 10(b). For example, if we place the third circle in the upper triangular region of the first drawing in Figure 10(a), we then need to reroute the upper horizontal edge around M and N and obtain drawing ② in Figure 10(b). ◀

4.2.0.1 Drawings of type $i \in \{1, 2, 3, 4\}$.

By Lemma 10, any simple tripartite-circle drawing of $K_{2,2,n}$ can be seen as an extension of one of the four drawings of $K_{2,2,0}$ in Figure 10(b). For example, Figure 9(a) displays a drawing of type 4 and Figure 9(b) a drawing of type 1. In our figures, we color the edges of $K_{2,2,0}$ red and the remaining $4n$ edges green. For an illustration consider Figure 11; note that some edges are omitted for clarity. We first count the number of crossings with red edges (i.e., the red-red and red-green crossings).

▶ **Lemma 11.** *In a simple tripartite-circle drawing D of $K_{2,2,n}$ with $n \geq 3$, the number of red-red and red-green crossings is at least*

$$\begin{cases} 2 \cdot (d_n(y_1, x_1) + d_n(x_2, y_2) + d_n(y_3, x_3) + d_n(x_4, y_4)) + 1 & \text{if } D \text{ is of type 1,} \\ 2n + 1 & \text{if } D \text{ is of type 2 or 3,} \\ 2 \cdot (d_n(y_1, x_1) + d_n(x_2, y_2)) + n & \text{if } D \text{ is of type 4.} \end{cases} \tag{8}$$

The number of green-green crossings is

$$f_n(x_1, x_2) + f_n(x_3, x_4) + f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4), \tag{9}$$

and the total number of crossings is the sum of (8) and (9).

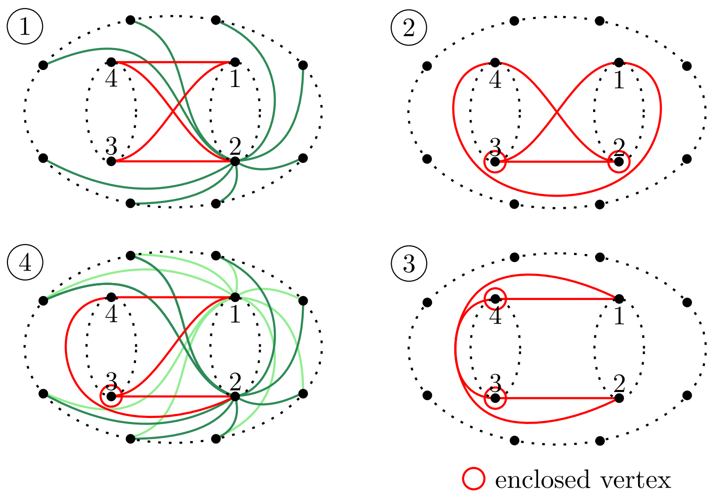


Figure 11 The four simple tripartite-circle drawings of $K_{2,2,n}$. Red edges connect vertices on the two inner circles; all other edges are green (some of them omitted for clarity). Drawings 2 and 3 have two enclosed vertices. A green edge from these vertices must cross a red edge.

Proof. There is exactly one red-red crossing for types 1, 2, and 3 and none for type 4. *Enclosed* vertices in drawings of types 2, 3, and 4 are those separated from the outer circle by red edges. The green edges incident to enclosed vertices must cross at least one red edge, so each enclosed vertex contributes at least n to the number of red-green crossings. For drawings of types 2 and 3, the two enclosed vertices guarantee at least $2n$ red-green crossings, and adding the 1 red-red crossing yields the claimed count.

Recall that we consider the vertices on the outer circle P in counterclockwise order. In a drawing of type 1, a green edge from vertex $i \in \{1, 3\}$ crosses two red edges if the other vertex lies in the interval $[y_i, x_i)$; otherwise it does not cross any red edges. Note that the number of these vertices is $d_n(y_i, x_i)$. Likewise, a green edge from a vertex $i \in \{2, 4\}$ crosses two red edges if the other vertex lies in the interval $[x_i, y_i)$. For type 1 drawings therefore the number of red-green crossings is at least $2 \cdot (d_n(y_1, x_1) + d_n(x_2, y_2) + d_n(y_3, x_3) + d_n(x_4, y_4))$. Adding the 1 red-red crossing yields the claimed count.

The same holds for green edges incident to vertices 1 or 2 in a drawing of type 4, so the number of red-green crossings in a type 4 drawing is at least $2 \cdot (d_n(y_1, x_1) + d_n(x_2, y_2)) + n$, where the n term counts red-green crossings in which the green edge is incident to the enclosed vertex 3.

We use Theorem 4 (and the note immediately following it) to count the green-green crossings as

$$f_n(x_1, x_2) + f_n(x_3, x_4) + f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4). \qquad \blacktriangleleft$$

Now to complete the proof of Lemma 9 and therefore of Theorem 3 it is sufficient to show that the sum of (8) and (9) is at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n - 3$ for any of the four types of drawings. By Equation (2), each term of (9) satisfies $f_n(a, b) \geq \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.

We start with two lemmas bounding some of the terms in (8) and (9). For notational convenience, we write $Z_i = d_n(y_{i+1}, y_i)$ and $z_i = \min\{d_n(y_{i+1}, y_i), d_n(y_i, y_{i+1})\} = \min\{Z_i, n - Z_i\}$ for $i = 1, 3$ (the minimum distance between vertices y_{i+1} and y_i on a circle with n vertices).

▶ **Lemma 12.** *For a simple tripartite-circle drawing of $K_{2,2,n}$ with $n \geq 3$, let $Z_i = d_n(y_{i+1}, y_i)$ for $i \in \{1, 3\}$. The following inequality holds:*

$$2d_n(y_i, x_i) + 2d_n(x_{i+1}, y_{i+1}) + f_n(x_i, x_{i+1}) \geq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1 - 2Z_i.$$

Moreover, the inequality can be strengthened in the following cases:

$$2d_n(y_i, x_i) + 2d_n(x_{i+1}, y_{i+1}) + f_n(x_i, x_{i+1}) \geq \begin{cases} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n - 1 - 2Z_i + 2n; & \text{if ccw order } \neq x_{i+1}y_{i+1}y_ix_i \\ \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n - 1 - 2Z_i + \left\lfloor \left(Z_i - \frac{n-2}{2}\right)^2 \right\rfloor & \text{if ccw order } = x_{i+1}y_{i+1}y_ix_i \\ & \text{and } Z_i \geq n/2 \end{cases}$$

Proof. Note that the (general) statement is equivalent to showing

$$2\big(d_n(y_i, x_i) + d_n(x_{i+1}, y_{i+1}) + d_n(y_{i+1}, y_i)\big) + f_n(x_i, x_{i+1}) \geq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1.$$

A short case analysis (by considering the six counterclockwise orders) verifies that for every four points a, b, c, d, the following holds:

$$d_n(a, b) + d_n(b, c) + d_n(c, d) = \begin{cases} d_n(a, d) & \text{if the ccw order is } abcd \\ d_n(a, d) + 2n & \text{if the ccw order is } adcb \\ d_n(a, d) + n & \text{otherwise.} \end{cases} \qquad (10)$$

With $a = x_{i+1}, b = y_{i+1}, c = y_i, d = x_i$, Equation (10) implies that

$$2\big(d_n(x_{i+1}, y_{i+1}) + d_n(y_{i+1}, y_i) + d_n(y_i, x_i)\big) + f_n(x_i, x_{i+1}) \geq 2d_n(x_{i+1}, x_i) + f_n(x_i, x_{i+1}). \qquad (11)$$

The right side of Inequality (11) can be expressed as the quadratic function

$$2d_n(x_{i+1}, x_i) + f_n(x_i, x_{i+1}) = d_n(x_{i+1}, x_i)^2 + (2 - n)d_n(x_{i+1}, x_i) + \binom{n}{2}, \qquad (12)$$

which is minimized for $d_n(x_{i+1}, x_i) = \lfloor \frac{n-2}{2} \rfloor$. Evaluation at $d_n(x_{i+1}, x_i) = \lfloor \frac{n-2}{2} \rfloor$ yields

$$2d_n(x_{i+1}, x_i) + f_n(x_i, x_{i+1}) \geq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1.$$

This finishes the proof of the general statement.

For the strengthening, we consider the two cases. If the counterclockwise order is different from $x_{i+1}y_{i+1}y_ix_i$, note that we can add at least a $2n$ term to the right side of the Inequality (11). If the counterclockwise order is $x_{i+1}y_{i+1}y_ix_i$ and $Z_i \geq n/2$, then $d_n(x_{i+1}, x_i) \geq d_n(y_{i+1}, y_i) = Z_i \geq n/2$ and the expression in Equation (12) is minimized for $d_n(x_{i+1}, x_i) = Z_i$. ◀

Now, we show a lower bound on the remaining four f_n-terms in (9). To do so, we define

$$\Delta_k := k \bmod 2.$$

Recall that $z_1 = \min\{d_n(y_2, y_1), d_n(y_1, y_2)\}$ and $z_3 = \min\{d_n(y_4, y_3), d_n(y_3, y_4)\}$.

▶ **Lemma 13.** *For a simple tripartite-circle drawing of $K_{2,2,n}$ with $n \geq 3$, let $S = f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4) - 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.*

i. *If $y_1, y_2 \in [y_3, y_4]$ or $y_1, y_2 \in [y_4, y_3]$, then it holds that*

$$S \geq z_1^2 + z_3^2 - \Delta_n \Delta_{z_1+z_3}$$

ii. *If $y_1 \in (y_3, y_4)$ and $y_2 \in (y_4, y_3)$ (or vice versa), then it holds that*

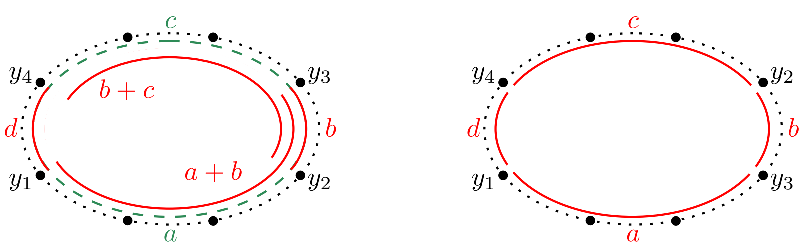
$$S \geq {}^1\!/\!_4 \left(z_1^2 + (n - z_1)^2 + z_3^2 + (n - z_3)^2\right) - {}^1\!/\!_2 \Delta_n.$$

Moreover, in all cases it holds that

$$S \geq z_1^2 - \Delta_n \Delta_{z_1}.$$

Proof. First note that exchanging y_1 and y_2 (or y_3 and y_4) does not influence $f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4)$. Consequently, by swapping y_1 and y_2 or y_3 and y_4, all counterclockwise orders can be transformed to one of the following two: $y_1y_2y_3y_4$ (non-alternating, i.e., Case 13.1) and $y_1y_3y_2y_4$ (alternating, i.e., Case 13.2)).

▶ Case 13.1. Without loss of generality, we consider the counterclockwise order y_1, y_2, y_3, y_4 and define $a := d_n(y_1, y_2)$, $b := d_n(y_2, y_3)$, $c := d_n(y_3, y_4)$, and $d := d_n(y_4, y_1)$, see also Figure 12(a). Then, it holds that



(a) Case 13.1, non-alternating. (b) Case 13.2, alternating.

Figure 12 Illustration of the proof of Lemma 13.

$$\begin{aligned}
f_n(y_1,y_3)+&f_n(y_1,y_4)+f_n(y_2,y_3)+f_n(y_2,y_4)\\
&=4\binom{n}{2}-(a+b)(n-(a+b))-d(n-d)-b(n-b)-(b+c)(n-(b+c))\\
&=2n^2-2n+a^2+b^2+c^2+d^2-(a+b+c+d)n+2b(a+b+c-n)\\
&=2n^2-2n+a^2+b^2+c^2+d^2-n^2-2bd\\
&=4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor-\Delta_n+a^2+c^2+(b-d)^2
\end{aligned}$$

The inequality $a^2+c^2+(b-d)^2 \geq z_1^2+z_3^2+\Delta_n\Delta_{z_1+z_3+1}$ holds when $a>z_1$, $c>z_3$, $b\neq d$, z_1+z_3 is odd, or n is even. At least one of these conditions holds as $a=z_1, c=z_3, b=d$, and z_1+z_3 even imply that $n=z_1+z_3+2b$ is also even. Hence,

$$S \geq z_1^2+z_3^2+\Delta_n\Delta_{z_1+z_3+1}-\Delta_n = z_1^2+z_3^2-\Delta_n\Delta_{z_1+z_3}$$

This finishes the proof of Case 13.1.

Note that $S \geq z_1^2+z_3^2-\Delta_n\Delta_{z_1+z_3}$ implies that $S\geq z_1^2-\Delta_n\Delta_{z_1}$ because $z_3^2-\Delta_n\Delta_{z_1+z_3}\geq 0\geq -\Delta_n\Delta_{z_1}$ when z_3 is odd, and $\Delta_{z_1+z_3}=\Delta_{z_1}$ when z_3 is even.

▶ Case 13.2. Without loss of generality, we consider the order y_1,y_3,y_2,y_4 and define $a:=d_n(y_1,y_3)$, $b:=d_n(y_3,y_2)$, $c:=d_n(y_2,y_4)$, and $d:=d_n(y_4,y_1)$, see also Figure 12(b). Then

$$\begin{aligned}
f_n(y_1,y_3)+&f_n(y_1,y_4)+f_n(y_2,y_3)+f_n(y_2,y_4)\\
&=4\binom{n}{2}-a(n-a)-d(n-d)-b(n-b)-c(n-c)\\
&=n^2-2n+a^2+b^2+c^2+d^2\\
&=4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor-\Delta_n+a^2+b^2+c^2+d^2
\end{aligned}$$

Note that $a^2+b^2\geq a^2+b^2-1/2\left((a-b)^2-\Delta_{a+b}\right)=1/2\left((a+b)^2+\Delta_{a+b}\right)$. Similarly, $c^2+d^2\geq 1/2\left((c+d)^2+\Delta_{c+d}\right)$, $b^2+c^2\geq 1/2\left((b+c)^2+\Delta_{b+c}\right)$, and $d^2+a^2\geq 1/2\left((d+a)^2+\Delta_{d+a}\right)$. We thus have

$$\begin{aligned}
S &\geq a^2+b^2+c^2+d^2-\Delta_n\\
&\geq 1/4\left((a+b)^2+(c+d)^2+(b+c)^2+(d+a)^2+\Delta_{a+b}+\Delta_{c+d}+\Delta_{b+c}+\Delta_{d+a}-4\Delta_n\right)\\
&= 1/4\left(z_1^2+(n-z_1)^2+z_3^2+(n-z_3)^2\right)+1/4\left(\Delta_{z_1}+\Delta_{n-z_1}+\Delta_{z_3}+\Delta_{n-z_3}-4\Delta_n\right)\\
&\geq 1/4\left(z_1^2+(n-z_1)^2+z_3^2+(n-z_3)^2\right)-1/2\Delta_n.
\end{aligned}$$

The last inequality is clear if n is even. When n is odd, then for $i=1,3$ one of z_i and $n-z_i$ is even and the other odd. Therefore, it holds that $1/4(\Delta_{z_1}+\Delta_{n-z_1}+\Delta_{z_3}+\Delta_{n-z_3}-4\Delta_n)=-1/2\Delta_n$. Hence, Case 13.2 is proved.

Finally, note that in Case 13.2 it holds that

$$S\geq a^2+b^2+c^2+d^2-\Delta_n\geq 1/2(z_1^2+(n-z_1)^2+\Delta_{z_1}+\Delta_{n-z_1})-\Delta_n\geq z_1^2-\Delta_n\Delta_{z_1}.$$

The last inequality follows from the fact that $n-z_1\geq z_1$ and either n is even or $(n-z_1)^2-z_1^2=n(n-2z_1)\geq 2$. This finishes the proof. ◀

Now, we are ready to prove the lower bound for each type of drawing. We begin with the types that are simpler to analyze.

▶ **Lemma 14.** *If D is a simple tripartite-circle drawing of $K_{2,2,n}$ of type 2, 3, or 4 with $n \geq 3$, then the number of crossings in D is at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n - 3$.*

Proof. For a drawing of type 2 or 3, by Equations (8) and (9) from Lemma 11, the number of crossings is at least

$$1 + 2n + f_n(x_1, x_2) + f_n(x_3, x_4) + f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4).$$

Using Equation (2), this expression is bounded below by $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + 1 > 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n - 3$, as desired. Note that drawings of these types do not attain the minimum number of crossings.

Next, we consider drawings of type 4. By Lemma 11 the number of crossings is

$$\begin{aligned} &n + 2(d_n(y_1, x_1) + d_n(x_2, y_2)) + f_n(x_1, x_2) + f_n(x_3, x_4) \\ &\quad + f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4). \end{aligned}$$

We show the lower bound by considering two cases for $Z_1 = d_n(y_2, y_1)$. In each of the cases, we use Equation (2) to bound $f_n(x_3, x_4)$.

In the first case, it holds that $Z_1 \leq (n-1)/2$. Then $z_1 = Z_1$ and by Lemmas 12 and 13, the number of crossings is at least

$$n + 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1 - 2Z_1 + Z_1^2 - \Delta_n \Delta_{Z_1}$$

and $2n - 1 - 2Z_1 + Z_1^2 - \Delta_n \Delta_{Z_1} = 2n - 2 + (Z_1 - 1)^2 - \Delta_n \Delta_{Z_1} \geq 2n - 3$. This shows the claim.

In the second case, it holds that $Z_1 \geq n/2$. Then $z_1 = n - Z_1$. Here we distinguish the subcases whether or not $[x_1, x_2] \subseteq [y_1, y_2]$. If $[x_1, x_2] \subseteq [y_1, y_2]$. Using the third inequality of Lemma 12, the number of crossings is at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor +$

$$\begin{aligned} &n + n - 1 - 2Z_1 + \left\lfloor \left(Z_1 - \frac{n-2}{2}\right)^2 \right\rfloor + (n - Z_1)^2 - \Delta_n \Delta_{n-Z_1} \\ &\geq 2n - 3 + \left\lfloor 1/8\left((4Z_1 - 3n)^2 + (n-4)^2\right) \right\rfloor \\ &\geq 2n - 3. \end{aligned}$$

If $Z_1 \geq n/2$ and $[x_1, x_2] \not\subseteq [y_1, y_2]$, then the second inequality of Lemma 12 shows that the number of crossings is at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor +$

$$\begin{aligned} n + n - 1 - 2Z_1 + 2n + (n - Z_1)^2 - \Delta_n \Delta_{n-Z_1} & \\ &\geq 2n - 1 + 2(n - Z_1) + (n - Z_1)^2 - 1 \\ &= 2n - 3 + (n - Z_1 + 1)^2 \\ &\geq 2n - 3. \end{aligned}$$

This finishes the proof for drawings of type 4. ◀

▶ **Lemma 15.** *If D is a simple tripartite-circle drawing of $K_{2,2,n}$ of type 1 with $n \geq 3$, then the number of crossings in D is at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n - 3$.*

Proof. By Lemma 11, drawings of type 1 have at least the following number of crossings

$$2(d_n(y_1,x_1)+d_n(x_2,y_2)+d_n(y_3,x_3)+d_n(x_4,y_4))+f_n(x_1,x_2)+f_n(x_3,x_4)$$
$$+f_n(y_1,y_3)+f_n(y_1,y_4)+f_n(y_2,y_3)+f_n(y_2,y_4)+1.$$

For $i=1,3$, recall $Z_i=d_n(y_{i+1},y_i)$, $z_i=\min\{Z_i,n-Z_i\}$, and let

$$B_i=\begin{cases}0 & \text{if } Z_i\le(n-1)/2,\\ 2n & \text{if } Z_i\ge n/2 \text{ and ccw order } \ne x_{i+1}y_{i+1}y_ix_i,\\ \left\lfloor\left(Z_i-\frac{n-2}{2}\right)^2\right\rfloor & \text{if } Z_i\ge n/2 \text{ and ccw order } = x_{i+1}y_{i+1}y_ix_i.\end{cases}$$

▶ Case 15.1. First, we consider the case where Lemma 13i) applies. By Lemma 12, Lemma 13i), and Equation (2), the number of crossings is at least

$$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2n-3+(1-2Z_1+z_1^2+B_1)+(1-2Z_3+z_3^2+B_3)-\Delta_n\Delta_{z_1+z_3}.$$

We need to prove that $(1-2Z_1+z_1^2+B_1)+(1-2Z_3+z_3^2+B_3)-\Delta_n\Delta_{z_1+z_3}\ge 0$, which follows from the following three inequalities.

- If $Z_i\le(n-1)/2$, then $z_i=Z_i$ and $B_i=0$. Thus

  $$1-2Z_i+z_i^2+B_i=1-2Z_i+Z_i^2=(Z_i-1)^2\ge 1$$

  unless $z_i=Z_i=1$.
- If $Z_i\ge n/2$ and ccw order $\ne x_{i+1}y_{i+1}y_ix_i$, then $z_i=n-Z_i$ and $B_i=2n$. Thus

  $$1-2Z_i+z_i^2+B_i=1-2Z_i+(n-Z_i)^2+2n=(n-Z_i+1)^2\ge 4$$

  because $Z_i\le n-1$.
- If $Z_i\ge n/2$ and ccw order $=x_{i+1}y_{i+1}y_ix_i$, then $z_i=n-Z_i$ and $B_i=\left\lfloor\left(Z_i-\frac{n-2}{2}\right)^2\right\rfloor$. Thus

  $$1-2Z_i+z_i^2+B_i=1-2Z_i+(n-Z_i)^2+\left\lfloor\left(Z_i-\frac{n-2}{2}\right)^2\right\rfloor=\left\lfloor\frac{(4Z_i-3n)^2+(n-4)^2}{8}\right\rfloor\ge 1$$

  unless $n=3,4$, or 5 and $z_i=n-Z_i=1$.

Note that $1-2Z_1+z_1^2+B_1=1-2Z_3+z_3^2+B_3=0$ if and only if $z_1=z_3=1$ and thus $(1-2Z_1+z_1^2+B_1)+(1-2Z_3+z_3^2+B_3)-\Delta_n\Delta_{z_1+z_3}=0$. In all other cases, $1-2Z_1+z_1^2+B_1$ and $1-2Z_3+z_3^2+B_3$ are both ≥ 0 and at least one of them is ≥ 1 so that $(1-2Z_1+z_1^2+B_1)+(1-2Z_3+z_3^2+B_3)-\Delta_n\Delta_{z_1+z_3}\ge 0$.

▶ Case 15.2. Now, we consider the case where Lemma 13ii) applies; note that this implies $n\ge 4$. Also, $z_i^2+(n-z_i)^2=Z_i^2+(n-Z_i)^2$ for $i=1,3$. By Lemma 12, Lemma 13ii), and Equation (2), the number of crossings is at least

$$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2n-3+\left(1-2Z_1+B_1+{}^1\!/\!_4(Z_1^2+(n-Z_1)^2-\Delta_n)\right)$$
$$+\left(1-2Z_3+B_3+{}^1\!/\!_4(Z_3^2+(n-Z_3)^2-\Delta_n)\right).$$

It is enough to prove that $1-2Z_i+B_i+{}^1\!/\!_4(Z_i^2+(n-Z_i)^2-\Delta_n)\ge 0$ for $i=1,3$ in each of the following three cases:

- If $Z_i \le (n-1)/2$. Then $B_i = 0$ and

$$
\begin{aligned}
1 - 2Z_i + B_i + {}^1\!/\!_4(Z_i^2 + (n-Z_i)^2 - \Delta_n)& \\
&= {}^1\!/\!_4((Z_i-2)^2 + (n-Z_i+2)^2 - \Delta_n) - 1 - n \\
&\ge \frac{1}{4}\left(\left(\frac{n-5}{2}\right)^2 + \left(\frac{n+5}{2}\right)^2 - \Delta_n\right) - 1 - n \\
&= \frac{n^2+25-2\Delta_n}{8} - 1 - n = \frac{(n-4)^2+1-2\Delta_n}{8} \ge 0.
\end{aligned}
$$

- If $Z_i \ge n/2$ and $B_i = 2n$. Then

$$
\begin{aligned}
1 - 2Z_i + B_i + {}^1\!/\!_4(Z_i^2 + (n-Z_i)^2 - \Delta_n)& \\
&= {}^1\!/\!_4((Z_i-2)^2 + (n-Z_i+2)^2 - \Delta_n) - 1 + n \\
&\ge \frac{1}{4}\left(\left\lfloor\frac{n}{2}\right\rfloor^2 + \left\lceil\frac{n}{2}\right\rceil^2 - \Delta_n\right) - 1 + n = \frac{n^2-\Delta_n}{8} - 1 + n \\
&\ge \frac{n^2-\Delta_n}{8} + 2 - n = \frac{(n-4)^2-\Delta_n}{8} \ge 0.
\end{aligned}
$$

- If $Z_i \ge n/2$ and $B_i = \left\lfloor \left(Z_i - \frac{n-2}{2}\right)^2 \right\rfloor$. Then

$$
\begin{aligned}
1 - 2Z_i + B_i + {}^1\!/\!_4(Z_i^2 + (n-Z_i)^2 - \Delta_n)& \\
&= 1 - 2Z_i + \left\lfloor\left(Z_i - \frac{n-2}{2}\right)^2\right\rfloor + {}^1\!/\!_4(Z_i^2 + (n-Z_i)^2 - \Delta_n) \\
&\ge \frac{1}{4}\left(\left\lfloor\frac{n}{2}\right\rfloor^2 + \left\lceil\frac{n}{2}\right\rceil^2 - \Delta_n\right) + 2 - n = \frac{n^2-\Delta_n}{8} + 2 - n = \frac{(n-4)^2-\Delta_n}{8} \ge 0.
\end{aligned}
$$

This finishes the proof of the lemma. ◀

Together, Lemmas 10, 14, and 15 yield the lower bound.

▶ **Lemma 9.** *For every integer $n \ge 3$,*

$$\mathrm{cr}_{③}(K_{2,2,n}) \ge 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2n - 3.$$

Proof. Let D be a tripartite-circle drawing of $K_{2,2,n}$ with the minimum number of crossings among all such drawings. By crossing-minimality, D is simple. By Lemma 10, either D is of type 1, and Lemma 15 applies, or D is of type 2, 3, or 4, and Lemma 14 applies. In all cases, D has at least $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2n - 3$ crossings. ◀

5 Conclusion

In this paper, we determined the exact value of the tripartite-circle crossing number of $K_{m,n,p}$ for all $m, n \le 2$. The natural next questions in this direction include determining $\mathrm{cr}_{③}(K_{m,n,p})$ for other small values of m and n. Of particular interest are the values $(m,n) = (1,3)$, $(1,4)$, $(1,5)$, $(2,3)$ because in these cases the exact value of $\mathrm{cr}(K_{m,n,p})$ is known but the exact value of $\mathrm{cr}_{③}(K_{m,n,p})$ is not.

Another natural extension of this work is to study k-partite-circle drawings for $k > 3$. In particular, what is the k-partite-circle crossing number of $K_{2,\ldots,2,n}$ for $k > 3$?

One implication of our result is that $\mathrm{cr}(K_{2,4,n}) - \mathrm{cr}_{③}(K_{2,2,n}) = 3$. It is an open problem whether there is a connection between drawings of $K_{2,4,n}$ and tripartite-circle drawings of $K_{2,2,n}$ that implies this similarity in the different crossing numbers.

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