On Erdős-Szekeres Maker-Breaker games*

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Abstract

The *Erdős-Szekeres Maker-Breaker game* is a two-player competitive game where both players alternately place points anywhere in the two dimensional Euclidean plane without overlapping, such that no three points are collinear. The first player (*Maker*) starts the game by placing her point and wants to obtain an empty convex polygon of a given size k such that the vertices of the polygon are chosen from these points and the second player (*Breaker*) wants to prevent it. We show that Maker wins the game for $k \leq 8$. We also present a winning strategy for Maker for any k in general when Maker is allowed to place $(1+\varepsilon)$ times more points (each round on average) in comparison to Breaker, for any $\varepsilon > 0$. Further, we address the models of the game for equilateral empty convex polygons in the plane and empty convex polygons in square grids.

Keywords and phrases Erdős-Szekeres games, Maker-Breaker games, Convex polygon, Convex holes, Computational geometry

Digital Object Identifier 10.57717/cgt.v5i2.73

Funding This work was supported by the Czech Science Foundation Grant no. 24-12046S.

1 Introduction

One of the most well-known problems in discrete geometry is the *Erdős-Szekeres Problem* [6]. In this problem, a positive integer k is given as input, and we compute the minimum required number of points in the plane such that at least k out of them are in convex position. The points must be placed in *general position*, i.e., no three points are collinear. Erdős and Szekeres showed that the answer is finite by proving that the number is bounded from above by a function exponential in k. Further Erdős [5] posed the problem of finding the minimum number of points in the plane in general position (no three points are collinear) such that k points out of them can be chosen as the vertices of a convex polygon that does not contain any other point from the point set inside it. The empty convex polygon with k vertices is referred to as a k*-hole*. By $H(k)$ we denote the minimum number of points such that any configuration of $H(k)$ points in general position contains a k-hole. Horton [10] demonstrated that there are arbitrarily large point sets that do not contain a 7-hole, in contrast to the finiteness of the previous question. For k-holes with $k < 7$, positive results are present in the literature showing $H(5) = 10$ [7] and $H(6) = 30$ [9].

Competitive games between two players to achieve a geometric structure were studied previously [8]. The natural competitive game arising from the Erdős-Szekeres problem is the endeavor of two players to achieve a k-hole for a given positive integer k by alternately placing points in a general position in the plane. Depending on the goal of the game, three variants of the two-player game spawn from the Erdős-Szekeres problem.

1. Both players want to obtain a k-hole (*Maker-Maker*). Whoever obtains the k-hole first is the winner.

* A conference version of this paper appeared in the proceedings of the 36th Canadian Conference on Computational Geometry (CCCG 2024).

2. Both players want to avoid a k-hole (*Avoider-Avoider*). Whoever obtains the k-hole first is the loser.
3. The first player wants to obtain a k-hole and the second wants to prevent it (*Maker-Breaker*).

Valla [12] posed the variant of the game where the winner is the first player who obtains a convex set of k points as an open problem in his thesis. Kolipaka and Govindarajan [11] studied the Avoider-Avoider variant. They proved that the game with $k = 5$ ends after round 9 and the second player wins.

Later Aichholzer et al. [1] simplified the original proof by Kolipaka and Govindarajan, and also introduced a different variant of the game with colors, referred to as *bi-chromatic* variant. Here, the players alternately place points in a general position in the plane, the first player placing red points and the second player placing blue points. Aichholzer et al. [1] showed the winning strategy for the second player for $k = 3$ for the Avoider-Avoider version, where the players try to avoid a monochromatic k-hole. Then they introduced the *Maker-Maker* variant of the bi-chromatic game where both players try to obtain a monochromatic k-hole. Besides considering the k-holes, they considered the non-convex *general holes* as well. The general hole of size k is an empty simple polygon with k vertices. Aichholzer et al. [1] also showed that the first player can win for $k = 5$ in 9 turns in the Maker-Maker variant. Further, they studied the *Maker-Breaker* variant of the bi-chromatic game where the first player (Maker) wants to obtain a k-hole with only *red* vertices, and the second player (Breaker) just wants to prevent it by placing blue vertices. For this variant, Aichholzer et al. proved that the Maker wins by placing 8 points of her color for a 5-hole and has a general winning strategy for general holes of any given k.

In this paper, we study the *Maker-Breaker* variant of the Erdős-Szekeres type game (*ESMB*) in the plane. For notational brevity, we name two players Alice and Bob. Both of them place one point in each round alternately on $\mathbb{R}^2$, maintaining the general position for all the points throughout the game. Alice tries to obtain a k-hole of a given size k. Alice wins the game if she can obtain a k-hole, and the game ends after a finite number of moves. Otherwise, we conclude that Bob wins the game if he can always restrict Alice from forming a k-hole. In our model, unlike the bi-chromatic variant, Alice can use any point of her choice to form the empty convex polygon of the desired size. To the best of our knowledge, this variant was not studied previously and was only pointed out as interesting and challenging for $k \geq 7$ by Aichholzer et al. [1].

We note that Alice can win the game for a given k in r rounds if there is an empty $(k-1)$-gon at the end of the $(r-1)$-st round. Alice can extend this existing hole by placing one point very closely without violating the convexity of the newly formed hole. Thus, it can be concluded from the existing literature proving $H(6) = 30$ [9] that Alice wins the game up to $k \leq 7$. But we show that the minimum number of points required for Alice to win is much less than $H(k)$. Further, we show Alice has a winning strategy for $k = 8$ even if $H(7)$ could be arbitrarily large [10]. Then we prove that Alice can win the game for any k if the ratio of the number of points placed by Alice and the points placed by Bob in each round is $(1 + \varepsilon)$, for any small $\varepsilon > 0$.

Then we address the question of obtaining equilateral holes. This question has not been considered before and only makes sense in terms of the game, as there could be arbitrarily large point sets such that all the distances of point pairs are different. We prove that Alice can obtain an equilateral k-hole for $k = 4$.

Finally, we address the variant of the game on grids. The Erdős-Szekeres problem has been extensively studied concerning the position of points approximating the integer lattice [4, 14].

We consider the Maker-Breaker game on a square grid of size $n \times n$. Both players must place their points at the gridpoints. Considering this constraint, in this version, the players are allowed to violate the general position requirement. We characterize the winning strategy of both players depending on the values of n and k.

We formally state the results as follows.

▶ **Theorem 1.** *Alice can win the ESMB game by obtaining a 7-hole from 15 points in the 8-th round.*

▶ **Theorem 2.** *Alice can win the ESMB game by obtaining an 8-hole from 25 points in the 12-th round.*

▶ **Theorem 3.** *Alice can win the ESMB game by obtaining a k-hole for any given positive integer k if the ratio of the points placed by Alice to the points placed by Bob is $(1+\varepsilon)$ for any $\varepsilon > 0$.*

▶ **Theorem 4.** *Alice wins the ESMB game for an equilateral 4-hole by obtaining it from 7 points in the 4-th round.*

▶ **Theorem 5.** *Let us consider the ESMB game on an $n \times n$ square grid, where both players have to place their points on one of the vertices of the grid, and they are allowed to place three or more collinear points. Alice can win the game by obtaining a k-hole for any positive integer $k > 2$, if and only if $n \geq \lceil \frac{k}{2} \rceil$.*

2 Winning strategies for Maker

We start with the formation of a 5-hole in the ESMB game, which is a winning strategy for Alice when $k = 5$. Let A_t and B_t denote the points placed by Alice and Bob in the t-th round of the game, respectively.

▶ **Lemma 6.** *Alice wins the game by obtaining a 5-hole from 7 points in the 4-th round.*

Proof. Alice trivially obtains a 3-hole with 3 points in the second round by placing A_2 in a non-collinear position with A_1 and B_1. Then, Bob must place a point that is not in a convex position with the other three to prevent Alice from winning in the immediate next round. Now Alice places A_3 in such a way that she creates one or more 4-holes. We can follow that one 4-hole remains in the plane irrespective of the placement of B_3. Alice extends it to a 5-hole by placing A_4 accordingly. ◀

Now we show the winning strategy of Alice in the ESMB game for $k = 6, 7,$ and 8. We slightly update the notation here. We assume that the game starts with a $(k-1)$-hole and Bob places his point as B_0, followed by Alice's point A_1, and so on.

▶ **Lemma 7.** *Starting with a 5-hole, Alice can obtain a 6-hole in the game within the next two rounds.*

Proof. We note that there are 5 chords of a 5-hole such that each divides the 5-hole into a 4-hole and 3-hole. Bob has to place B_0 in the common intersection of all the 4-holes to prevent Alice from winning in the immediate next round. Since there are 5 such 3-holes, Alice can find two 4-holes intersecting only in one edge, after the placement of B_0. One instance is depicted in Figure 1, where we start with the 5-hole $ABCDE$ and $\overline{B_0E}$ is the common edge between two 4-holes ABB_0E and $CDEB_0$. Alice places A_1 in such a way that it creates two 5-holes intersecting in one triangle ($\triangle B_0EA_1$ in the figure). If Bob places his point inside this triangle, then two 5-holes remain in the plane. Namely $ABB_0B_1A_1$ and EB_1B_0CD for the instance in Figure 1. Otherwise, Bob places his point inside only one of the two 5-holes. Thus, after placement of B_1, Alice extends the remaining 5-hole to a 6-hole in the next round. ◀

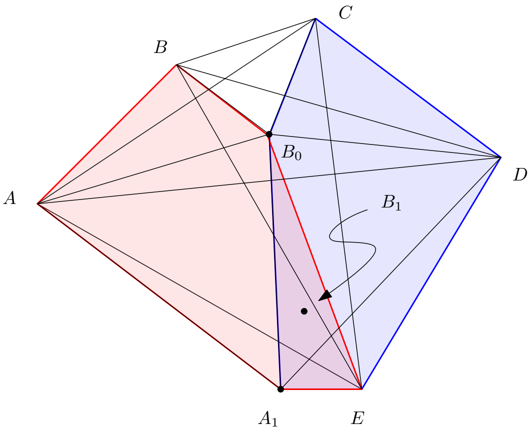

Figure 1 Formation of a 6-hole.

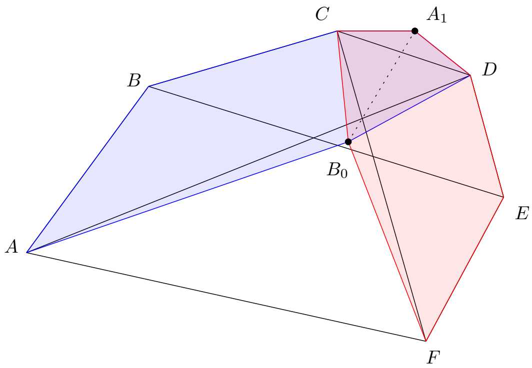

Figure 2 Formation of a 7-hole.

▶ **Lemma 8.** *Starting with a* 6*-hole, Alice can obtain a* 7*-hole in the game within the next two rounds.*

Proof. A 6-hole has three chords such that each of them divides the 6-hole into two 4-holes. An instance is depicted in Figure 2. $ABCDEF$ is the initial 6-hole and $\overline{AD}, \overline{BE}$, and $\overline{CF}$ are the chords dividing it into 4-holes. Since Bob can place B_0 inside the intersection of at most three out of these six 4-holes, Alice can find two 5-holes intersecting in one triangle after the placement of B_0 inside the 6-hole. She can extend these two 5-holes into two 6-holes intersecting in one convex quadrilateral. As a result, after the placement of B_1, at least one 6-hole remains in the plane that can be extended to a 7-hole in the next round. Figure 2 depicts the case where B_0 is inside three 4-holes namely $ADEF$, $ABCF$ and $BCDE$. Thus, Alice creates two 6-holes $ABCA_1DB_0$ and CA_1DEFB_0 by placing A_1. Bob will try to place B_1 in the intersection of these two 6-holes formed after placement of A_1, but B_1 can be placed inside at most one of the two triangles $\triangle B_0CA_1$ or $\triangle B_0DA_1$. Even if Bob chooses one of them to place B_0 inside, either $B_1A_1DEFB_0$ or $ABCA_1B_1B_0$ remains a 6-hole, ensuring the formation of a 7-hole in the next round by placing A_2 accordingly. The other cases arising from different placements of B_0 are analogous, considering the symmetry of the chords of the initial 6-hole. ◀

Combining Lemma 6, 7, and 8 we get the following theorem.

▶ **Theorem 1.** *Alice can win the ESMB game by obtaining a* 7*-hole from* 15 *points in the* 8*-th round.*

We note that in the cases of obtaining k-holes for $k = 6$ and 7, we considered the chords that divided the $(k-1)$-hole into half, where the size of the half was $k-2$. As a result, after placement of B_0, at least two $(k-2)$-holes remain in the plane. That gives Alice a chance to create two $(k-1)$-holes intersecting in a triangle or a convex quadrilateral. This observation is not true when we start with a 7-hole and as a result, we cannot ensure Alice's winning within the next 2 rounds. We prove Alice needs 5 more rounds to obtain an 8-hole starting with a 7-hole.

▶ **Lemma 9.** *Starting with a* 7*-hole, Alice can obtain an* 8*-hole in the game within the next five rounds.*

Proof. We begin with a similar approach as Lemma 8 that there are 7 chords in a 7-hole such that each divides the hole into one 5-hole and one 4-hole. The chords are depicted in Figure 3 as $\overline{AD}, \overline{BE}, \overline{CF}, \overline{DG}, \overline{EA}, \overline{FB}$, and $\overline{GC}$ in the 7-hole $ABCDEFG$. To play optimally, Bob places B_0 inside the 7-hole. After the placement of B_0, Alice can place A_1 in such a way that there are two 6-holes in the plane intersecting in one triangle. One instance is depicted in Figure 3 (left) with the two 6-holes namely AB_0EA_1FG and $BCDEA_1B_0$. If both 6-holes remain after placement of B_1, Alice extends these two 6-holes into two 7-holes by a similar strategy in Lemma 8 and wins in the following round. Thus, Bob must place B_1 inside at least one of these two 6-holes. Furthermore, Bob does not place B_1 inside the octagon $ABCDEA_1FG$ such that five vertices of the octagon are lying on the same side of the line passing through B_0 and B_1, as this will create a 7-hole. Thus, we have the following observation.

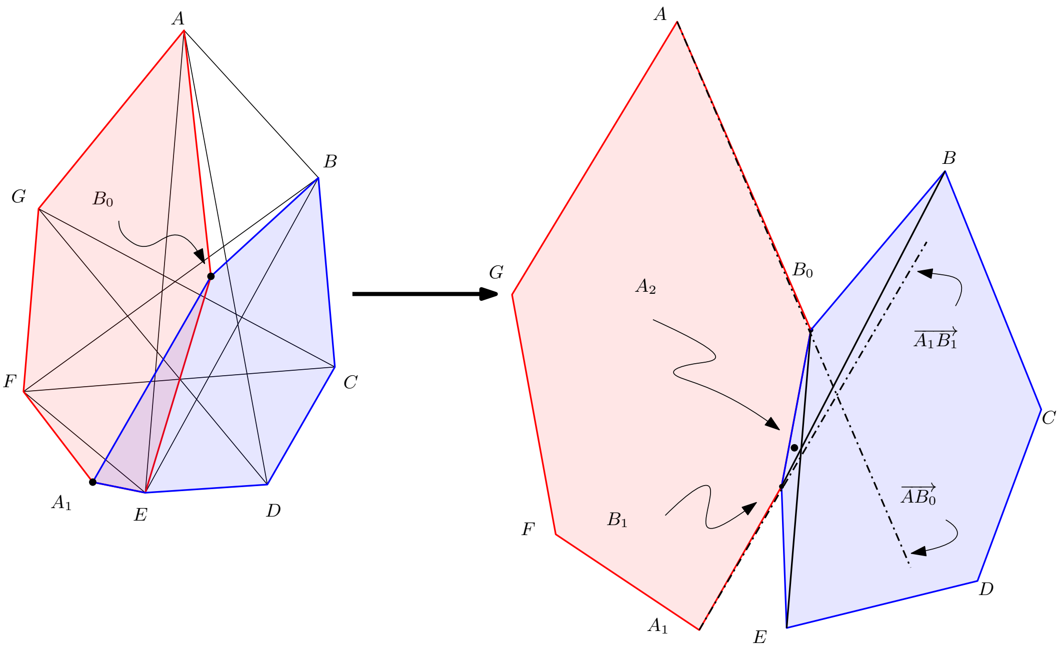

Figure 3 Observation 1.

▶ **Observation 1.** *After the placement of B_1, there are two* 6*-holes in the plane sharing exactly one common edge. Alice can place A_2 in such a way that it creates two* 6*-holes and one* 7*-hole. Moreover, both* 6*-holes share exactly one edge with the* 7*-hole, and these two shared edges are adjacent to each other in the* 7*-hole.*

As mentioned in Observation 1, the precise position of A_2 is inside the intersection of the triangles $\triangle B_0BB_1$, $\triangle B_0EB_1$ and the triangle formed by the sides $\overline{B_0B_1}, \overrightarrow{AB_0}$ and $\overrightarrow{A_1B_1}$.

Here $\overrightarrow{AB}$ denotes the prolongation of the segment $\overline{AB}$ from B. Now Bob must place B_2 inside the 7-hole to prevent Alice from winning in the immediate next round. Thus, after placement of B_2, we have the following observation.

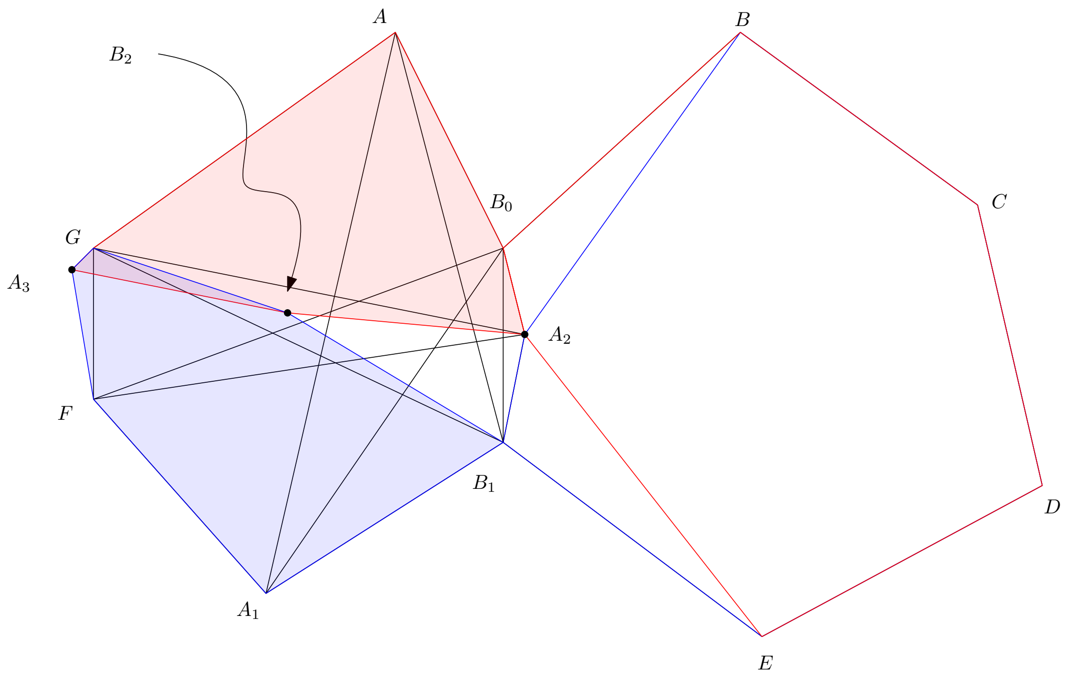

Figure 4 Observation 2.

▶ **Observation 2.** *Alice can place A_3 in such a way that after the placement of B_3, there are two 6-holes in the plane sharing exactly one vertex.*

Proof of Observation. Note that Bob does not place B_2 in such a way that there are two 6-holes sharing only one vertex, as this will help Alice to create two 7-holes sharing only one edge and win immediately. Moreover, there will be two 6-holes involving the vertices of the 7-hole (containing B_2 and B_3) following the same argument as Observation 1. If Alice can force the placement of B_3 such that the line passing through B_2 and B_3 intersects either $\overline{B_0A_2}$ or $\overline{A_2B_1}$ then Observation 2 holds. Thus, to force such placement of B_3, Alice places A_3 in such a way that there are two 6-holes intersecting in one triangle such that none of them contains both $\overline{B_0A_2}$ and $\overline{A_2B_1}$ as their edges. One instance is depicted in Figure 4. △

Using Observation 2, Alice extends both 6-holes to two disjoint 7-holes by placing A_4 accordingly, ensuring the formation of an 8-hole in the next round. The placement of A_4 for the instance considered in Figure 4 is depicted in Figure 5. ◀

Thus, by combining Theorem 1 and Lemma 9, we have the following theorem.

▶ **Theorem 2.** *Alice can win the ESMB game by obtaining an 8-hole from 25 points in the 12-th round.*

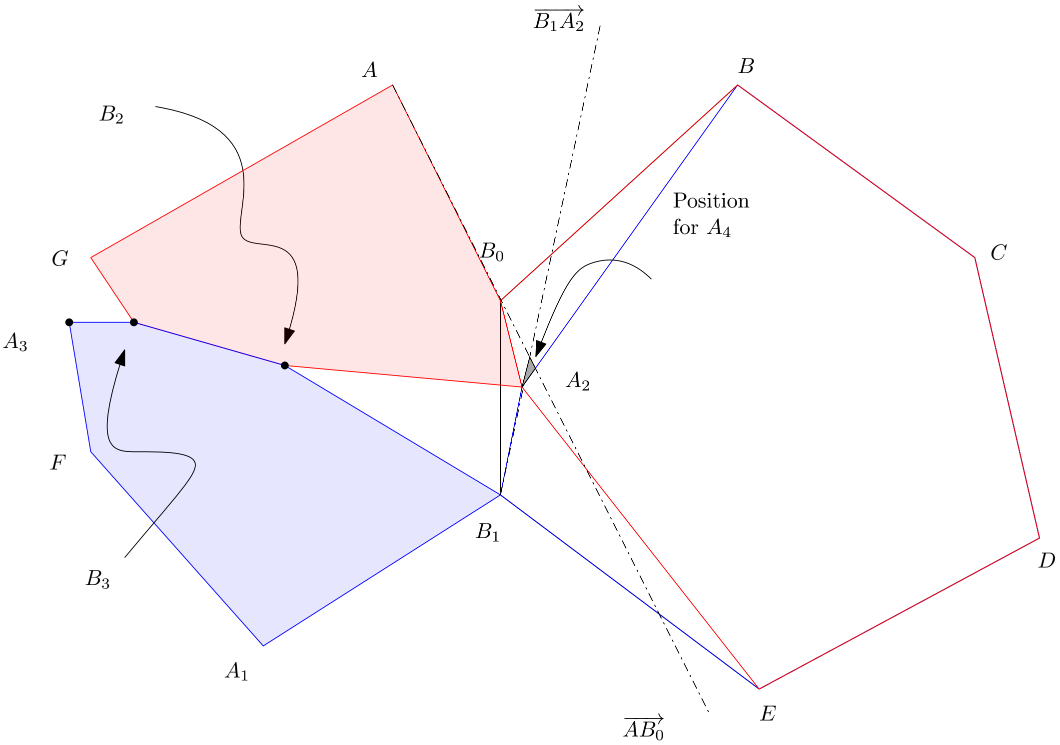

Figure 5 Placement of A_4 to obtain an 8-hole.

2.1 Winning strategy for Maker in general with a higher speed

Now we present a general strategy for Alice to win the game when she benefits with a higher speed than Bob. First, we assume that in each round, Alice places two points while Bob places only one. We show that Alice can obtain any k-hole for any given k in 2^{k-1} rounds.

▶ **Lemma 10.** *Alice can win a Maker-Breaker version of the Erdos-Szekers game by obtaining a k-hole in 2^{k-1} rounds for any given k if she is allowed to place two points in each round while Bob is allowed to place only one point each round.*

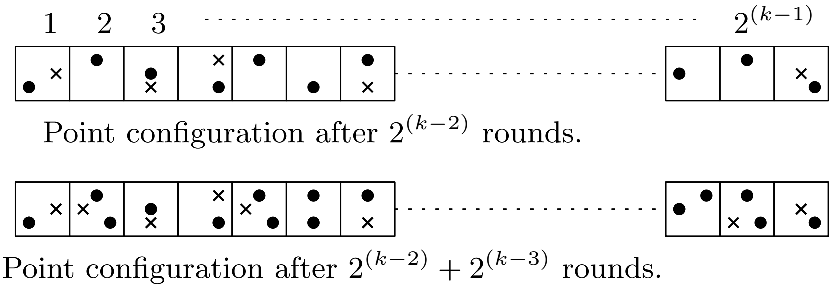

Figure 6 Game configurations when Alice has double the speed of Bob.

Proof. We prove the lemma by describing the strategy of Alice. Consider 2^{k-1} disjoint unit squares in the plane. Alice places two points inside two different empty squares in each round for the first 2^{k-2} rounds. In the next 2^{k-3} rounds, she places her points only in the squares where there are no points of Bob inside. She can always find 2^{k-2} squares since Bob can only place 2^{k-2} points in the first 2^{k-2} rounds. In the following 2^{k-4} rounds, she places her points into the squares that do not contain any points of Bob. Following the strategy, she can keep placing her points only in the squares that do not contain Bob's points, and as a result, she can place them into a convex position, achieving a k-hole in the 2^{k-1}-th round. ◀

Now we generalize the idea where the ratio of the points placed by Alice and Bob is $(1+\varepsilon)$ for any small $\varepsilon > 0$. In other words, Alice places at least one point more than Bob after the r-th round of the game, i.e., $\varepsilon = 1/r$. Then, using an argument similar to that of Lemma 10, we can conclude that after a finite number of steps, Alice can secure at least one such square that is free from a point of Bob. Thus, she iterates the strategy to achieve one square containing only k points placed by her. This takes $(r+1)^{k-1}$ turns to win the game. It is worth noting that the result holds asymptotically. Even if Alice is not permitted to place her points for a finite number of rounds, or Bob places more points than Alice for a finite number of rounds; given the benefit Alice has on the ratio as the number of rounds increases, Alice still wins the game. Hence, we have the following theorem.

▶ **Theorem 3.** *Alice can win the ESMB game by obtaining a k-hole for any given positive integer k if the ratio of the points placed by Alice to the points placed by Bob is $(1+\varepsilon)$ for any $\varepsilon > 0$.*

3 ESMB game for equilateral holes

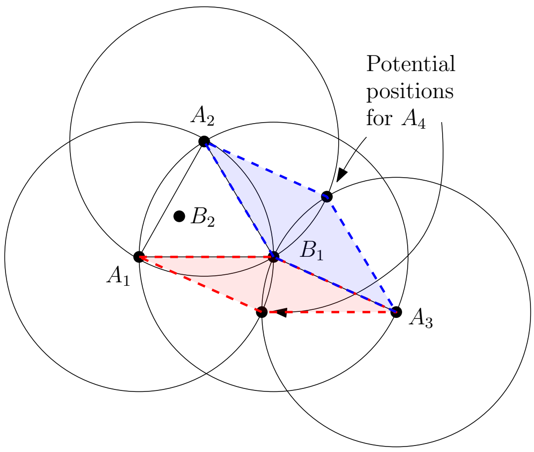

Figure 7 Formation of an equilateral 4-hole.

In this section, we address the question of obtaining equilateral holes for Maker. This question does not arise in the case of the classical Erdős-Szekeres problem: it is trivial to generate a point set of any size where no two pairs of points have the same distance between them.

We prove that Alice can create an equilateral 4-hole in the game. We describe the strategy in this subsection. It can be seen that Alice can create an equilateral triangle in the plane by placing A_2 in the second round, and Bob must place his point inside the triangle to prevent Alice from winning in the immediate next round. Moreover, Bob ensures that B_2 is placed in such a way that it is not equidistant from two points in $\{A_1, B_1, A_2\}$. In the third round, Alice places A_3 dividing one outer angle ($\angle A_1B_1A_2$ in the Figure 7) of the triangle in such a way that the length of $\overline{A_3B_1}$ is the same as the length of $\overline{A_1B_1}$. Moreover, the circle centered at A_3 with a radius of the same length as the length of $\overline{A_3B_1}$ intersects both circles centered at A_1 and A_2 of the same radius. These two intersection points act as two potential candidates for A_4 such that either $A_1A_4A_3B_1$ or $A_2A_4A_3B_1$ becomes an equilateral 4-hole depending on the placement of B_3. This gives the following result.

▶ **Theorem 4.** *Alice wins the ESMB game for an equilateral* 4*-hole by obtaining it from* 7 *points in the* 4*-th round.*

4 ESMB game on a square grid

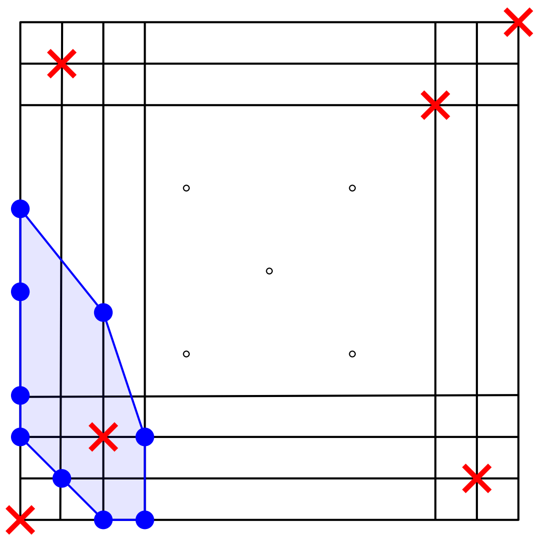

Figure 8 Winning strategy for Bob on a grid of $n \times n$ with $n < \lceil \frac{k}{2} \rceil$.

In this subsection, we study the game in a square grid of a fixed size, say $n \times n$. We show that if we allow the players to violate the general position assumption of the points, then Alice can win if and only if $n \geq \lceil \frac{k}{2} \rceil$. If the grid is of size $\lceil \frac{k}{2} \rceil$, it is easy to follow that Alice can ignore the placement of Bob and can form a k-hole from two consecutive rows or columns of the grid. But, interestingly, the bound is tight as Bob can prevent Alice from winning if $n < \lceil \frac{k}{2} \rceil$.

▶ **Lemma 11.** *Alice does not have a winning strategy for the ESMB game on an* $n \times n$ *square grid if* $n < \lceil \frac{k}{2} \rceil$ *with* $k > 2$.

Proof. We note that Alice cannot form a convex k-hole only by using the points from two consecutive rows or columns. Hence, she has to use points from at least three rows or columns. Since any convex polygon of size k must contain two points of at least one diagonal of the grid, Bob places his n points alternately on both diagonals depending on the placements of the points by Alice, shown in Figure 8. This prohibits Alice from obtaining a hole of the desired size. ◀

Thus, we have the following theorem.

▶ **Theorem 5.** *Let us consider the ESMB game on an* $n \times n$ *square grid, where both players have to place their points on one of the vertices of the grid, and they are allowed to place three or more collinear points. Alice can win the game by obtaining a* k*-hole for any positive integer* $k > 2$*, if and only if* $n \geq \lceil \frac{k}{2} \rceil$.

5 Conclusion

For Maker with a slightly higher speed than Breaker, we have presented a general strategy to win the ESMB game, but it takes exponentially long to finish. Also, we observe that for small k like 8, the game finishes much faster. Thus, it is an intriguing open question to address whether there exists a winning strategy for Maker with the same speed as Breaker, even if there are constructions of the large sets without hole [4, 10, 13, 14]. An important observation is that the *Horton sets* [10], which are the building blocks of large point sets without k-holes for $k > 7$, are *fragile* in the sense that inserting one unwanted point in the set can create an unwanted hole. Moreover, the expected number of holes in a random point set of size n selected from a convex shape of unit area in the plane is $O(n^2)$ [2]. On the other hand, the existing results on the expected size of the largest hole in random point sets [3] are logarithmic in terms of the number of points. Therefore, can Breaker delay the game infinitely by preventing Maker from forming a k-hole? If so, it is also interesting to study the maximum value of k for which Maker can always win.

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