Bounds on the Crossing Number of m-Fold Symmetric Configurations of Points in the Plane

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Abstract

For any finite set of points P in general position in the plane, we consider the drawing of the complete graph with vertex set P, whose edges are the straight-line segments joining pairs of points. The *crossing number* of P is the number of pairs of segments that intersect in their interior. The minimum crossing number over all such sets P of n points is known as the *rectilinear* or *geometric crossing number* of the complete graph K_n. The problem of determining this crossing number was posed by Erdős and Guy in the early 1970s and, although it has been extensively studied, it remains open to this date with exact values known only for $n \leq 27$ and $n = 30$. In 2010, it was conjectured that whenever the number of points is a multiple of 3, there are crossing optimal point sets with 3-fold (triangular) rotational symmetry. Motivated by this conjecture, this paper provides lower and upper bounds for sym-$\overline{\text{cr}}_m(K_n)$, the rectilinear crossing number of K_n when the minimum is restricted to m-fold symmetric configurations of points in the plane. Our bounds coincide for even m, which surprisingly provides the exact value

$$\text{sym-}\overline{\text{cr}}_m(K_n) = \frac{1}{2}\binom{n}{4} + \frac{1}{2}\binom{n/2}{2} + \frac{n^2}{6}\binom{m/2-1}{2}.$$

For odd m, our bounds give

$$\text{sym-}\overline{\text{cr}}_m(K_n) = \left(\frac{1}{2} - \frac{m-1}{2m^2} + \Theta\left(\frac{1}{m^3}\right)\right)\binom{n}{4} + \Theta_m(n^3).$$

Keywords and phrases Crossing number, rectilinear crossing number, complete graph, rotational symmetry, m-fold symmetry

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1 Introduction

In a *drawing* of a simple graph G (on the plane), the vertices of G are represented by points in the plane, and the edges are curves joining the vertices. We assume that the edges are never tangent to each other and that the interior of any edge does not contain any vertices. Two edges *cross* if they intersect in their interior (sharing an endpoint is not considered a crossing). If more than two edges cross at the same point, we count the crossing with multiplicity; that is, if k edges cross at the same point, this point represents $\binom{k}{2}$ crossings. The number of crossings in a drawing D of a graph is denoted by $\text{Cr}(D)$. The minimum number of crossings over all drawings of a graph G is denoted by $\text{cr}(G) := \min\{\text{Cr}(D) : D \text{ drawing of } G\}$.

In the 1940s, Paul Turán [43] considered this problem for complete bipartite graphs and due to its history [23] the problem is now known as the *Brick Factory Problem.* Independently, Harary and Hill studied the problem for complete graphs [37] and conjectured that

$$\operatorname{cr}(K_n) = H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

This conjecture remains open; however, in the last 10 years a great deal of progress has been made [2, 3, 17, 20]. These results were based on tools that were first developed for a related problem, where the minimum number of crossings is taken over a smaller family of drawings. Namely, those known as *rectilinear* or *geometric drawings* of G, where the set of vertices is in general position (no three points on a line) and the edges are straight line segments. In this context, we define the *rectilinear* or *geometric crossing number* of a graph G, denoted by $\overline{\operatorname{cr}}(G)$, as the minimum number of crossings over all rectilinear drawings of G.

The problem of determining the rectilinear crossing number of the complete graph, $\overline{\operatorname{cr}}(K_n)$ (see Figure 1a), was posed by Erdős and Guy in the early 1970s [29, 34] and has been extensively studied since then [1, 10, 12, 18, 19, 20, 21]. It is equivalent to finding the minimum number of convex quadrilaterals determined by n points, because two edges in a rectilinear drawing cross if and only if they are the diagonals of a convex quadrilateral, see Figure 2. Since a rectilinear drawing D of a complete graph is determined by the location of its set of vertices P, we typically write $\operatorname{Cr}(P)$ instead of $\operatorname{Cr}(D)$. Note that $\operatorname{cr}(K_n) \le \overline{\operatorname{cr}}(K_n)$; in fact $\operatorname{cr}(K_n) < \overline{\operatorname{cr}}(K_n)$ for $n \ge 8$ [8, 34, 35, 40]. The exact value of $\overline{\operatorname{cr}}(K_n)$ is only known for $n \le 27$ and $n = 30$ [6, 15, 18, 19, 24, 25, 35]. An important tool that allowed the determination (or verification) of these values for $n \le 18$ is a comprehensive database of *order types* with up to 11 points [18, 19, 22, 39]: two labeled point sets have the same *order type* when the corresponding triples have the same orientation (which determines 4-tuples in convex position). The database is maintained online [16] and is a great resource for experimentation in combinatorial geometry.

In this paper, we consider configurations of points with *(rotational) m-fold symmetry*, that is, sets P for which there is a $\frac{2\pi}{m}$-rotation $\mathcal{R}$ such that $\mathcal{R}(P) = P$ ($m = 2$ corresponds to central symmetry). In this case, the *m-rotational orbit* of a point $x \in P$ is the set of points $[x] := \{x, \mathcal{R}(x), \mathcal{R}^2(x), \ldots, \mathcal{R}^{m-1}(x)\} \subseteq P$. We study the rectilinear crossing number of K_n when the minimum is restricted to vertex sets of n points with rotational m-fold symmetry (see Figure 1bc). This crossing number is denoted by sym-$\overline{\operatorname{cr}}_m(K_n)$. Our main contribution is establishing the exact value of this crossing number, sym-$\overline{\operatorname{cr}}_m(K_n)$, when m is even (Theorem 4), and providing strong upper and lower bounds when m is odd (Theorem 5).

The latest improvements on the quest for determining $\overline{\operatorname{cr}}(K_n)$ use the following approach: Let P be a set of n points in general position in the plane (no 3 points on a line). For an integer $0 \le k \le n/2$, we say that a pair of points $\{p, q\} \subseteq P$, or equivalently the segment pq, is a *k-edge* of P if the line spanned by p and q separates exactly k points of P from the remaining $n-k-2$ points, see Figure 3. Note that any pair of points in P is a k-edge for some k and that the condition that $k \le n/2$ always points to the lighter side of the corresponding segment. For example, any segment in Figure 3c separates 2 points of P from the remaining 4. Then each segment is a 2-edge and not a 4-edge. An *at-most-k-edge* is a j-edge for some $j \le k$. We denote the number of k-edges and at-most-k-edges of P by $E_k(P)$ and $E_{\le k}(P)$, respectively.

A strongly related concept is that of *k-sets*: A set $Q \subseteq P$ is a k-set of P if there is a straight line that separates Q from $P \setminus Q$. Rotating a separating line for a k-set until it hits a point on each side yields a $(k-1)$-edge, and this correspondence is bijective, so

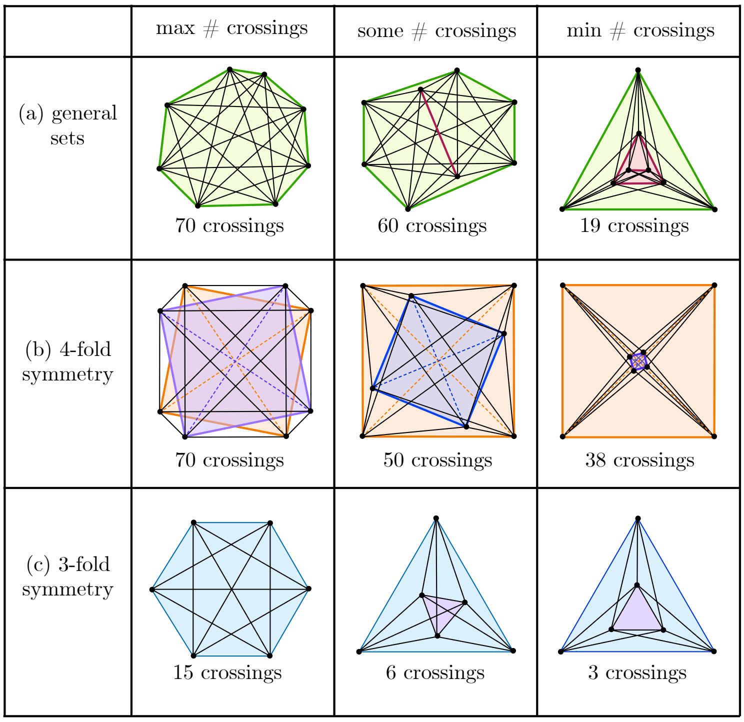

Figure 1 Comparing crossing numbers. Any set of n points in convex position determines $\binom{n}{4}$ crossings, the maximum possible as shown in the first column. (a) Sets of $n = 8$ points, the third drawing is crossing optimal with $\overline{\text{cr}}(K_8) = 19$ crossings. (b) Sets of 8 points with 4-fold symmetry, the third set is crossing optimal under the 4-fold symmetry restriction with sym-$\overline{\text{cr}}_4(K_8) = 38$ crossings. (c) Sets of $n = 6$ points, the third set is crossing optimal (with or without the 3-fold symmetry restriction) with sym-$\overline{\text{cr}}_3(K_6) = \overline{\text{cr}}(K_6) = 3$ crossings.

counting k-sets is equivalent to counting $(k-1)$-edges. The notions of k-sets and k-edges go back to early work by Erdős, Lovász, Simmons, and Straus [30], who introduced these parameters in the 1970s. Beyond their intrinsic role in extremal combinatorial geometry, k-edges (equivalently k-sets) are a standard proxy for the “combinatorial complexity” of a point configuration: they appear in range searching [13, 26, 42], geometric optimization [27, 28, 41], and in algorithmic contexts where one needs to understand the evolution of separations under motion or rotation (see, for example, [14, 36, 38] for applications to motion planning algorithms in robotics).

In our setting, their importance is amplified by the fact that the crossing number of a set of points P can be expressed in terms of a weighted sum of its numbers of k-edges (Theorem 1). The key is to count pairs of edges that do not cross: fix an oriented edge $\overrightarrow{pq}$ and note

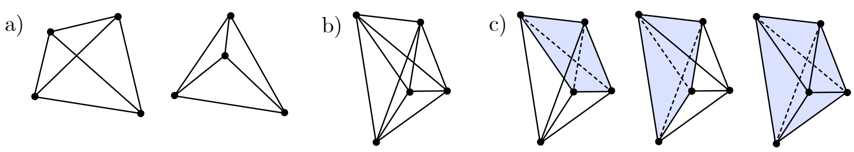

Figure 2 (a) The two essentially different rectilinear drawings of K_4: convex position (a crossing) and not convex position (no crossing), respectively. (b) A rectilinear drawing of K_5 with 3 crossings. (c) Each crossing in part (b) is identified with a (shaded) convex quadrilateral.

that p with any point to the right of $\overrightarrow{pq}$ and q with any point to the left of $\overrightarrow{pq}$ form a pair of non-crossing edges. Hence, the k-edges keep count of how many pairs of non-crossing edges are associated to each edge. Summing the appropriate weights over all k-edges and subtracting from the total number of pairs of edges produces an exact expression for the total number of crossings (and an equivalent form in terms of $E_{\leq k}(P)$).

▶ **Theorem 1** (Crossing number identity [8, 40])**.** *Let D be a rectilinear drawing of the complete graph K_n with set of vertices P. Then*

$$\mathrm{Cr}(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2\rfloor-1} k(n-2-k)E_k(P)$$
$$= \left(\sum_{k=0}^{\lfloor n/2\rfloor-2} (n-2k-3)E_{\leq k}(P)\right) - \frac{3}{4}\binom{n}{3} + (1+(-1)^{n+1})\frac{1}{8}\binom{n}{2}.$$

The relevance of this connection between $E_{\leq k}(P)$ and $\overline{\mathrm{cr}}(K_n)$ became evident when the authors [8] and, independently, Lovász et al. [40] proved that $E_{\leq k}(P) \geq 3\binom{k+2}{2}$, which together with Theorem 1 provided a significant improvement to the previously best known lower bound for $\overline{\mathrm{cr}}(K_n)$. Since then, improvements to this crucial inequality on the number of at-most-k-edges have automatically provided better lower bounds on $\overline{\mathrm{cr}}(K_n)$. For a broader overview of the landscape around the problem of determining $\overline{\mathrm{cr}}(K_n)$, including its connection to at-most-k edges, halving lines, and k-sets, we refer to the survey by the authors and Salazar [12]. In contrast to the original problem, there is no conjectured value for $\overline{\mathrm{cr}}(K_n)$. The closest approximation is a conjecture in [5] stating that (if 3 divides n) the rectilinear crossing number is achieved by sets with 3-fold symmetry; and a bound for sets known as *3-decomposable*. Since then, the study of the rectilinear crossing number for sets with rotational symmetry gained interest. Recently, the case where the minimum is restricted to the family of centrally symmetric drawings (2-fold symmetry), denoted by $\overline{\mathrm{cr}}_{\mathrm{cs}}(K_{2n})$, was considered in [7]. There, they proved the following result, which is probably the first identity for the rectilinear crossing number of a large family of rectilinear drawings of K_n.

▶ **Theorem 2** (Centrally symmetric identity [7])**.** *For any even positive integer n, we have*

$$\overline{\mathrm{cr}}_{\mathrm{cs}}(K_n) = 2\binom{n/2}{4} + \binom{n/2}{2}^2 = \frac{1}{2}\binom{n}{4} + \frac{1}{2}\binom{n/2}{2}.$$

Their proof was based on Theorem 1 together with the following tight bound on the number of at most k-edges.

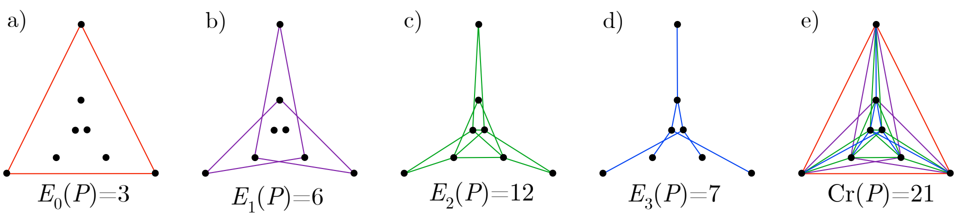

Figure 3 Five copies of a set P of 8 points in general position. The first four copies show the k-edges of P for $k = 0, 1, 2,$ and 3, respectively. The last copy shows the complete graph with vertex set P and its crossing number. Here $E_{\leq 0}(P) = E_0(P) = 3, E_{\leq 1}(P) = E_0(P) + E_1(P) = 3 + 6 = 9, E_{\leq 2}(P) = E_0(P) + E_1(P) + E_2(P) = 3 + 6 + 12 = 21$, and $E_{\leq 3}(P) = E_0(P) + E_1(P) + E_2(P) + E_3(P) = 3 + 6 + 12 + 7 = 28 = \binom{8}{2}$, which is the number of edges in K_8.

▶ **Lemma 3** (Lemma 2.1 in [7]). *For any centrally symmetric set P of n points in general position in the plane, n even, and for any $k < n/2 - 1$,*

$$E_{\leq k}(P) \geq 4\binom{k+2}{2}.$$

Our main tool is a sequence of m-fold symmetric analogues of the standard "at-most-k-edge" lower bounds. Concretely, we prove a recursive inequality describing how $E_{\leq k}(P)$ changes when one removes an entire rotational orbit on the convex hull (Lemma 6); this in turn yields our crucial k-edge bound for m-fold symmetric sets (Lemma 7). Combined with the crossing number identity (Theorem 1), this immediately implies a new lower bound for the crossing number under symmetry constraints.[1] A striking feature is that the behavior of the crossing number differs substantially depending on the parity of m. When m is even, we show that this lower bound is tight by an explicit construction presented in Section 3.2, settling the crossing number of symmetric configurations with even symmetry as follows (note that Theorem 4 agrees with Theorem 2 when $m = 2$):

▶ **Theorem 4.** *For positive integers m and n such that m is even and $m|n$, we have*

$$\text{sym-}\overline{\text{cr}}_m(K_n) = \frac{1}{2}\binom{n}{4} + \frac{1}{2}\binom{n/2}{2} + \frac{n^2}{6}\binom{m/2-1}{2}.$$

The situation for odd m is more complicated. We first improve the bound in Lemma 7 for $k \geq \left(\frac{m-1}{m}\right)\frac{n}{2}$ (Lemma 9). This improvement relies on the use of *allowable sequences*, introduced by Goodman and Pollack in the 1980's [31, 32, 33], which allow for a tighter comparison of the k-edges of a set and the set without an orbit (Lemma 8). Based on the central argument in [6], we further improve Lemma 9 for $k > \frac{n}{2} - \frac{n}{6m}$ (Lemma 13). These improvements on the crucial lemma (see Figure 4) together with Theorem 1 yield our lower bound for odd symmetry. We complement this bound with an explicit construction, presented in Section 4.2, that matches the lower bound up to the $O(1/m^2)$ term in the leading coefficient. This results in the two-sided bounds in the next theorem.

▶ **Theorem 5.** *For positive integers $m \geq 3$ and n such that m is odd and $m|n$, we have*

$$\text{sym-}\overline{\text{cr}}_m(K_n) \geq \left(\frac{1}{2} - \frac{m-1}{2m^2} - \frac{97m-31}{270m^3(m+1)}\right)\binom{n}{4} + \Theta(n^3)$$

and for $m \geq 5$,

$$\text{sym-}\overline{\text{cr}}_m(K_n) \leq \left(\frac{1}{2} - \frac{m-1}{2m^2}\right)\binom{n}{4} + \Theta(n^3).$$

It is customary to normalize by $\binom{n}{4}$ (rather than by n^4). This is because, as stated above, each crossing in a rectilinear drawing of K_n is determined by a unique set of four vertices in convex position. Thus $\overline{\text{cr}}(K_n)/\binom{n}{4}$ can be interpreted as the *density of convex 4-tuples*.

The paper is organized as follows: In Section 2, we present our crucial lemma and its improvements for odd symmetry (Lemmas 7, 9, and 13). We then focus on even symmetry in Section 3, where we prove Theorem 4 starting with the lower bound in Section 3.1 using Lemma 7 and Theorem 1; followed by the matching upper bound given by our construction

[1] We explicitly provide this process only for even m in Section 3.1 because the rest of our work improves the corresponding bound for odd m.

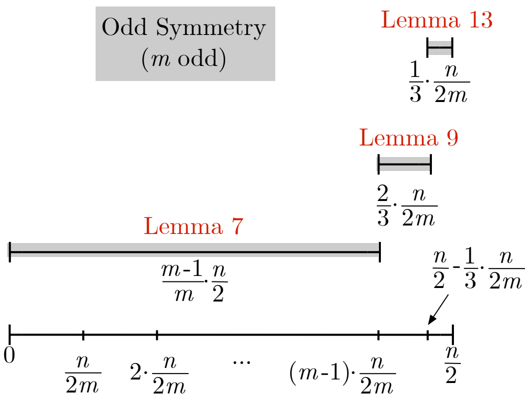

Figure 4 The crucial lemmas for odd rotational symmetry (m odd). Lemmas 7 and 9 coincide when $0 \le k \le \left(\frac{m-1}{m}\right)\frac{n}{2}$ but Lemma 9 is better for larger k. Lemma 13 is better than Lemma 9 for k close to $\frac{n}{2}$.

in Section 3.2. In Section 4, we move on to odd symmetry, where we prove Theorem 5: The lower bound in Section 4.1 follows from Lemmas 9 and 13 together with Theorem 1; and the construction in Section 4.2 settles the upper bound. In Section 5, we conclude by presenting a discussion on how our bounds for odd symmetry could potentially be improved and proposing some open questions.

2 Crucial lemma: lower bounds on at-most-k-edges

This section develops the main technical tool used throughout the paper: lower bounds on the number of at-most-k edges determined by an m-fold symmetric point set. Recall that our ultimate goal is to lower-bound the crossing number via the crossing number identity (Theorem 1), which expresses $\mathrm{Cr}(P)$ as a weighted sum of the quantities $E_{\le k}(P)$. Thus, any improvement on $E_{\le k}(P)$ immediately translates into a stronger lower bound on sym-$\overline{\mathrm{cr}}_m(K_n)$.

Our starting point is a recursive inequality that compares $E_{\le k}(P)$ with the corresponding quantity after removing one full rotational orbit on the convex hull (Lemma 6). Iterating this inequality yields an explicit baseline bound for all m (Lemma 7), which is tight for even m and forms the backbone of the proof of Theorem 4.

When m is odd, the baseline bound can be sharpened in two ranges of k. In Section 2.1, we use circular/allowable sequences to capture additional transpositions that are not accounted for by the orbit-removal recursion, leading to a mid-range improvement (Lemma 9). In Section 2.2 we adapt the central-range approach of [6] to obtain our strongest bound for k close to $n/2$ (Lemma 13). Figure 4 summarizes how these bounds compare and which one is used in each range.

▶ **Lemma 6.** *Let P be a set of n points in general position in the plane and $0 \le k \le \lceil n/2 \rceil - 2$ an integer. Suppose P is m-fold symmetric and consider a point x in the convex hull of P. Let $P' = P - [x]$, where $[x] \subset P$ is the m-rotational orbit of x. Then*

$$E_{\le k}(P) \ge E_{\le k - \lceil m/2 \rceil}(P') + 2m(k+1) - m \cdot \min(k+1, \lceil m/2 \rceil - 1).$$

(Note that $E_{\le j}(P) = 0$ for negative values of j.)

Proof. Let P' be the set obtained from P by removing the m points of $[x]$. Note that P' is an m-fold symmetric set of $n-m$ points. Consider an edge e of P (any line passing through two points of P) and let ℓ be the line parallel to e passing through the center of symmetry of P. Note that ℓ divides $[x]$ in almost half, leaving at most $\lceil m/2\rceil$ points on each side. Since the smaller side of e is contained on one of the two sides of ℓ, then e leaves at most $\lceil m/2\rceil$ points of $[x]$ on its smaller side. This means that if e' is a j-edge of P', then e' is an at-most-$(j+\lceil m/2\rceil)$-edge of P. Therefore, for any $k\leq\lceil n/2\rceil-2$ any at-most-$(k-\lceil m/2\rceil)$-edge of P' is an at-most-k-edge of P. And none of these at-most-k-edges is incident on points of $[x]$.

Because the points of $[x]$ are on the convex hull of P, each of them participates in exactly two j-edges for each $0\leq j\leq\lceil n/2\rceil-2$. This gives at most $2m(k+1)$ at-most-k-edges of P incident on points of $[x]$. However, some of these edges may be incident to two points of $[x]$: as many as $m(k+1)$ when $k\leq\lceil m/2\rceil-1$ and as many as $m(\lceil m/2\rceil-1)$ (all pairs of $[x]$, except its halving lines when m is even) when $k\geq\lceil m/2\rceil-1$. In summary, there are at least $E_{\leq k-\lceil m/2\rceil}(P')$ at-most-k-edges of P not incident on $[x]$, and at least $2m(k+1)-m\cdot\min(k+1,\lceil m/2\rceil-1)$ at-most-k-edges of P incident on $[x]$. ◀

▶ **Lemma 7.** *Let m be a positive integer. For any set P of n points with m-fold symmetry and for any $0\leq k\leq\lceil n/2\rceil-2$, we have that*

$$E_{\leq k}(P)\geq\begin{cases}4\left(1-\frac{1}{m+1}\right)\left(\binom{k+2}{2}-\binom{r+1}{2}\right)+mr & \text{if } m \text{ is odd},\\ 4\left(\binom{k+2}{2}-\binom{r+1}{2}\right)+mr & \text{if } m \text{ is even},\end{cases}\tag{1}$$

where $k+1=r \bmod \lceil m/2\rceil$.

Proof. If $0\leq k\leq\lceil m/2\rceil-2$, then $r=k+1$ and $\min(k+1,\lceil m/2\rceil-1)=k+1$. Then, by Lemma 6, $E_{\leq k}(P)\geq 0+m(k+1)=mr$ matching the right-hand-side of (1).

If $k=\lceil m/2\rceil-1$, then $r=0$. By Lemma 6,

$$\begin{aligned}E_{\leq k}(P)&\geq 0+2m(k+1)-m\cdot\min(k+1,k)=m(k+2)\\ &=\frac{2m}{k+1}\binom{k+2}{2}=\frac{2m}{\lceil m/2\rceil}\binom{k+2}{2}=\begin{cases}4\left(\frac{m}{m+1}\right)\binom{k+2}{2} & \text{if } m \text{ is odd},\\ 4\binom{k+2}{2} & \text{if } m \text{ is even},\end{cases}\end{aligned}$$

matching the right-hand-side of (1).

We prove the result by induction on n. The inequality holds for $n=m$ because a k-edge of a set of m points is an at-most-$(\lceil m/2\rceil-1)$-edge (proved above). Now let $n\geq m$ and assume that the inequality holds for any m-fold symmetric set of n points and any $k<\lceil n/2\rceil-2$. Consider any m-fold symmetric set P of $n+m$ points. Let x be a point on its convex hull and P' be the set obtained from P by removing the m points of $[x]$. Note that P' is an m-fold symmetric set of n points and so it satisfies the induction hypothesis. We already proved the result holds for $k\leq\lceil m/2\rceil-1$. Assume that $k\geq\lceil m/2\rceil$. Note that $(k-\lceil m/2\rceil)+1\equiv k+1\equiv r \pmod{\lceil m/2\rceil}$. By Lemma 6,

$$\begin{aligned}E_{\leq k}(P)&\geq E_{\leq k-\lceil m/2\rceil}(P')+2m(k+1)-m\cdot\min(k+1,\lceil m/2\rceil-1)\\ &\geq m\left(2k+3-\left\lceil\frac{m}{2}\right\rceil\right)+\begin{cases}4\left(\frac{m}{m+1}\right)\left(\binom{k+2-\lceil m/2\rceil}{2}-\binom{r+1}{2}\right)+mr \text{ if } m \text{ is odd},\\ 4\left(\binom{k+2-\lceil m/2\rceil}{2}-\binom{r+1}{2}\right)+mr \text{ if } m \text{ is even}.\end{cases}\end{aligned}\tag{2}$$

Since

$$\binom{k+2-\lceil m/2\rceil}{2} = \binom{k+2}{2} - \frac{1}{2}\left\lceil \frac{m}{2}\right\rceil \left(2k+3-\left\lceil \frac{m}{2}\right\rceil\right),$$

then the right-hand-side of (2) equals

$$\begin{cases} 4\left(\frac{m}{m+1}\right)\left(\binom{k+2}{2}-\binom{r+1}{2}\right) + mr \\ \qquad + \left(2k+3-\lceil\frac{m}{2}\rceil\right)\left(m-2\cdot\frac{m}{m+1}\cdot\frac{m+1}{2}\right) & \text{if } m \text{ is odd,} \\ 4\left(\binom{k+2}{2}-\binom{r+1}{2}\right) + mr + \left(2k+3-\lceil\frac{m}{2}\rceil\right)\left(m-2\cdot\frac{m}{2}\right) & \text{if } m \text{ is even,} \end{cases}$$

$$= \begin{cases} 4\left(1-\frac{1}{m+1}\right)\left(\binom{k+2}{2}-\binom{r+1}{2}\right) + mr & \text{if } m \text{ is odd,} \\ 4\left(\binom{k+2}{2}-\binom{r+1}{2}\right) + mr & \text{if } m \text{ is even.} \end{cases}$$

◀

2.1 Odd symmetry: mid-range improvement of the crucial lemma

In this section, we improve Lemma 7 for odd m using circular sequences (or more generally, allowable sequences), which were introduced by Goodman and Pollack in the 1980s [31, 32, 33]). We use the terminology and observations in [4, Section 2]. In short, the combinatorial properties of a set P of n points in the plane are encoded by its *circular sequence*, a doubly-infinite periodic sequence of permutations of the set $\{1, 2, \ldots, n\}$. These permutations record the order in which the points of P are projected onto a directed line that is continuously rotated (counterclockwise) in the plane. If P is in general position (no three points on a line), then the order changes only at finitely many different directions, and each change corresponds to a set of pairwise disjoint transpositions of adjacent elements (points whose projections coincide). In general, several such transpositions may occur at the same direction (for example, when multiple pairs span parallel lines).

A 180°-rotation of the line is enough to generate the entire sequence. This corresponds to a *halfperiod* of the circular sequence and is generated by exactly $\binom{n}{2}$ transpositions, one per pair of points in P, occurring at h different directions for some $h \leq \binom{n}{2}$. We typically write such halfperiod as an $(h+1) \times n$ arrangement, whose rows are the ordered permutations and a row is obtained from the previous row by some transpositions of consecutive elements (Figure 5-left shows the circular sequence of a regular 9-gon, it has halfperiod of length $h = 9$). A transposition involving the elements in positions k and $k+1$ or in positions $n-k$ and $n-k+1$ (positions k and $k+1$ from the right) of a permutation is called a *k-transposition*.

In this context, $(k+1)$-transpositions correspond to k-edges. The lower bound in Lemma 7, counts $\leq (k+1)$-transpositions of two types: those involving at least one of the m points that are removed during the induction and those that are $(\leq k+1-(m+1)/2)$-transpositions in P'. However, there might also be $(k+1-(m-1)/2)$-transpositions in P' that are $(k+1)$-transpositions in P, and such $\leq (k+1)$-transpositions have not been counted by the lower bound in Lemma 7. We show that when m is odd and k is large enough, such transpositions always exist. More precisely,

▶ **Lemma 8.** *If m is odd and $\frac{m-1}{2} \cdot \frac{n}{m} \leq k \leq \lceil \frac{n}{2} \rceil - 2$, then there are at least $m(k+1) - \frac{m-1}{2}n$ different $(k+1)$-transpositions of P that are $(k+1-\frac{m-1}{2})$-transpositions of P'.*

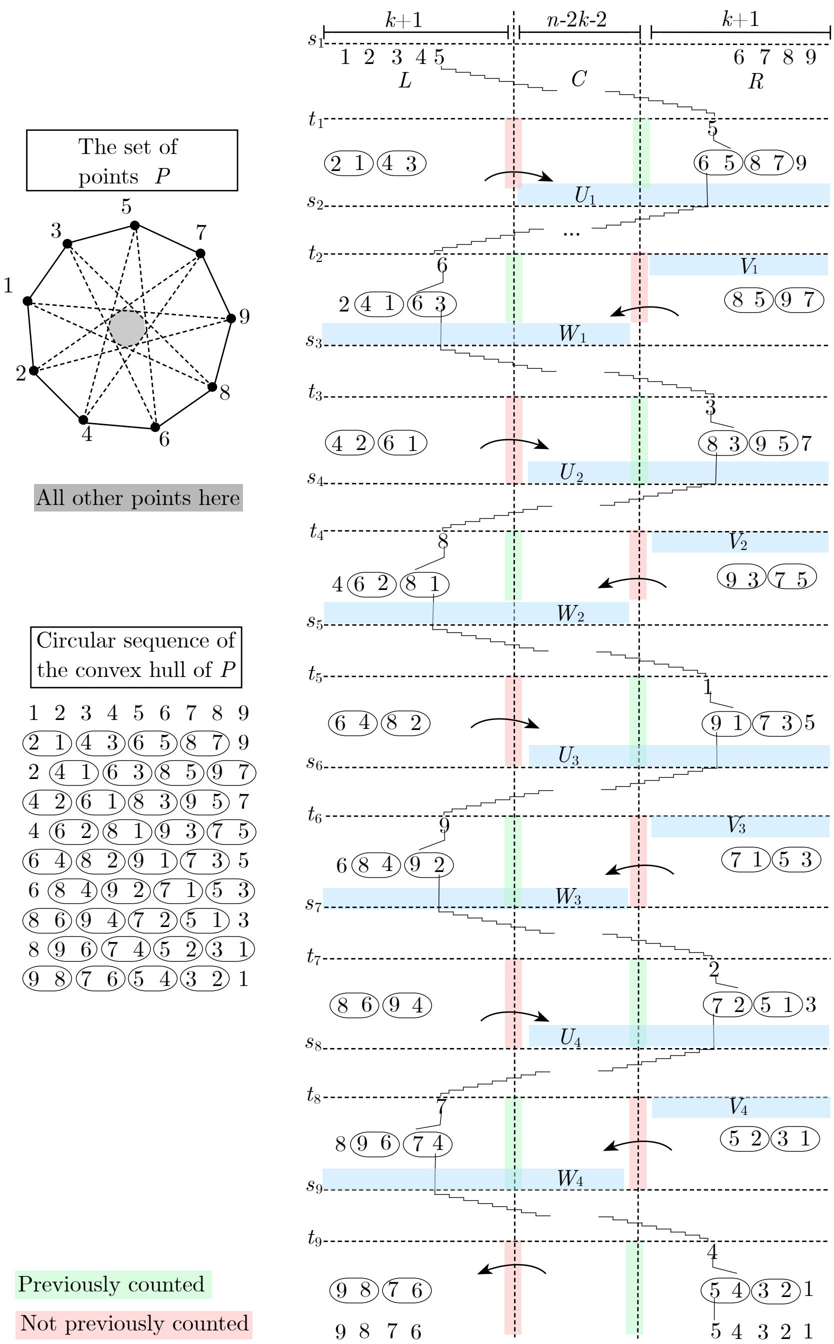

Figure 5 The structure of a 9-fold symmetric circular sequence to illustrate the description in the proof of Lemma 8. (Left) The convex hull ∂P of the set P is a regular 9-gon, the remaining points of P are in the center of the star formed by its main diagonals. A halfperiod of the circular sequence of ∂P is included. The circled pairs indicate the transpositions that occurred from the previous permutation. (Right) A description of a halfperiod of the circular sequence of P, including the start and end of the permutations obtained by transpositions of elements of ∂P. Each of these elements transposes with the remaining $n-9$ points in consecutive permutations moving across the circular sequence between permutations (rows) s_i and t_i for some $1 \le i \le 9$.

Proof. Denote by ∂P the set of points of P in the boundary of its convex hull. Consider any point $x \in \partial P$. By an argument similar to that in [4, Lemma 4], we can modify P without increasing the number of at-most-k-edges and preserving the m-fold symmetry by moving x (together with its orbit) away from the center of P along a line through x that halves the rest of the set P and is not parallel to any other connecting line. Moving x far enough guarantees that ∂P consists of exactly m points (the orbit of x) and that the rest of the points of P are in the center of the star formed by the halving lines of ∂P. Moreover, we can assume that the rest of the points are as far as needed from ∂P so that all transpositions of any element of ∂P with the elements of $P - \partial P$ occur consecutively on the circular sequence of P. Let $\pi = \{\pi_0, \pi_1, \pi_2, \ldots, \pi_h\}$ be a half-period of the circular sequence of P, with $\pi_0 = (1, 2, 3, \ldots, n)$ showing the points of ∂P in its first $(m+1)/2$ and last $(m-1)/2$ positions and such that the first $n - m$ transpositions of π are transpositions of the point $(m+1)/2$ with the points in $P - \partial P$ (see Figure 5). Then there exist $1 = s_1 \leq t_1 \leq s_2 \leq t_2, \ldots, \leq s_m \leq t_m$ such that each permutation in $\{\pi_{s_i}, \pi_{s_i+1}, \ldots, \pi_{t_i}\}$ is obtained by a transposition with the i^{th} element of ∂P along the cycle (star corresponding to the circulant of ∂P with jump $(m+3)/2$) with vertices $(m+1)/2, (m+3)/2, (m-3)/2, (m+7)/2, \ldots, 1, m, 2, m-2, 4, m-4, \ldots, (m-1)/2$ if $m \bmod 4 = 1$ or $(m+1)/2, (m+3)/2, (m-3)/2, (m+7)/2, \ldots, m, 1, m-1, 3, m-3, \ldots, (m-1)/2$ if $m \bmod 4 = 3$.

Partition the columns of π into 3 regions: the left region L formed by the first $k+1$ elements of each row, the central region C formed by the next $n - 2(k+1)$ elements of each row, and the right region R formed by the last $k+1$ elements of each row. This partition of π inherits a partition of row r into the sets $L(r)$, $C(r)$, and $R(r)$. For $1 \leq i \leq (m-1)/2$, let

$$
\begin{aligned}
U_i &= L(s_{2i-1}) \cap (C(t_{2i}-1) \cup R(t_{2i}-1)),\\
V_i &= V_{i-1} \cap R(t_{2i}), \text{ where } V_0 = R(1),\\
W_i &= V_i \cap (L(s_{2i+1}-1) \cup C(s_{2i+1}-1)), \text{ and } W_0 = \emptyset.
\end{aligned}
$$

Because an element of V_{i-1} that is not in V_i either belongs to W_{i-1} or must be replaced by one of the $n - 2(k+1)$ elements of $C(t_{2i-1})$ or by an element of U_i, then

$$|V_i| \geq |V_{i-1}| - |W_{i-1}| - (n - 2(k+1)) - |U_i|.$$

Since $|V_0| = k + 1$,

$$|V_{(m-1)/2}| \geq (k+1) - \frac{m-1}{2} \cdot (n - 2(k+1)) - \sum_{i=1}^{(m-3)/2} (|U_i| + |W_i|) - |U_{(m-1)/2}|.$$

Finally, note that each element of U_i is involved in a k-transposition from row j to $j+1$ for some $t_{2i-1} \leq j < s_{2i}$ and by taking the last such transposition (so that it involves exactly one element of U_i), we have a total of at least $|U_i|$ different k-transpositions in that range. Similarly, there are at least $|W_i|$ different k-transpositions from row j to $j+1$ for some $t_{2i} \leq j < s_{2i+1}$. (Although this holds for $i = (m-1)/2$, we only use it for $1 \leq i \leq (m-3)/2$ in order to avoid double counting transpositions in the next argument.) Finally, any element of $V_{(m-1)/2}$ is involved in a k-transposition involving elements in positions $n-k-1$ and $n-k$ from some row j to $j+1$ for some $t_{m-1} \leq j < s_m$ or in a k-transposition involving elements in positions $k+1$ and $k+2$ from some row j to $j+1$ for some $j \geq t_m$. Therefore, the number of desired k-transpositions is at least

$$|V_{(m-1)/2}| + \sum_{i=1}^{(m-3)/2} (|U_i| + |W_i|)| + |U_{(m-1)/2}| \geq (k+1) - \frac{m-1}{2} \cdot (n - 2(k+1)) = m(k+1) - \frac{m-1}{2}n.$$

◀

Our new lower bound is a direct application of Lemma 8.

▶ **Lemma 9.** *Let $m \geq 3$ be odd. For any set P of n points with m-fold symmetry and for any $0 \leq k \leq \lceil n/2 \rceil - 2$,*

$$E_{\leq k}(P) \geq 4\left(\frac{m}{m+1}\right)\left(\binom{k+2}{2} - \binom{r+1}{2}\right) + mr + m\binom{k - \frac{m-1}{2}\cdot\frac{n}{m} + 2}{2}$$

where $k+1 = r \bmod \lceil m/2 \rceil$.

Proof. A $(k+1)$-transposition in Lemma 8 corresponds to a k-edge of P. Then, when $k \geq \frac{m-1}{2} \cdot \frac{n}{m}$, the lower bound for $E_{\leq k}(P)$ in Lemma 7 for m odd increases by

$$\sum_{j=\frac{m-1}{2}\cdot\frac{n}{m}}^{k} \left(m(j+1) - \frac{m-1}{2}n\right) = \sum_{j'=0}^{k-\frac{m-1}{2}\cdot\frac{n}{m}} \left(m(j' + \frac{m-1}{2}\cdot\frac{n}{m} + 1) - \frac{m-1}{2}n\right)$$

$$= m \sum_{j'=0}^{k-\frac{m-1}{2}\cdot\frac{n}{m}} (j'+1) = m\binom{k - \frac{m-1}{2}\cdot\frac{n}{m} + 2}{2}$$

◀

2.2 Odd symmetry: central-range improvement of the crucial lemma

The equivalent version of the previous lemma for arbitrary point sets was shown to be tight for $0 \leq k \leq 4n/9$ in [6]. In that same work, the authors proved an inequality for at-least-k edges. As the name suggests, the number of at-least-k edges of a point set P is defined as $E_{\geq k}(P) = \binom{n}{2} - E_{\leq (k-1)}(P) = \sum_{j=k}^{\lfloor n/2 \rfloor - 1} E_j(P)$. With this new result they were able to further improve the lower bound for $E_{\leq k}(P)$ on the range $4n/9 \leq k \leq n/2$. In this section, we follow the same strategy to improve our lower bound for m-fold symmetric sets. We first show how we can generate a recursive sequence (u_k) of improved bounds beginning with an already established bound $u_{b-1} \leq E_{\leq b-1}(P)$ for a suitable value b. Next, we show how to estimate these new bounds u_k, and we apply these results together with Lemma 9 from the previous subsection. We use the following result from [6], which we have reworded for point sets:

▶ **Lemma 10** (Theorem 1 in [6]). *Let P be a set of n points and $\pi = (\pi_0, \pi_1, \ldots, \pi_h)$ be a halfperiod of the circular sequence defined by P. For every integer k, $1 \leq k < n/2$, define $C(k, \pi_j)$ as the set of elements in the middle $n - 2k$ positions of π_j, and*

$$s(k,\pi) = \min\left\{|C(k,\pi_0) \cap C(k,\pi_i)| : 0 \leq i \leq \binom{n}{2}\right\}.$$

If $s = s(k, \pi)$, then

$$E_{\geq k}(P) \leq (n - 2k - 1) E_{k-1}(P) - \frac{s}{2}(E_{k-1}(P) - n + 1).$$

The following result gives improved lower bounds for the range $b \leq k \leq (n-3)/2$, provided that the initial lower bound $u_{b-1} \leq E_{b-1}(P)$ is good enough.

▶ **Lemma 11** (Generalization of Theorem 2 in [6]). *Let $n/4 \leq b-1 \leq (n-3)/2$. For each $k \geq b-1$, define the following recursive sequence: the initial term u_{b-1} is a lower bound of $E_{\leq(b-1)}(P)$; that is, $u_{b-1} \leq E_{\leq(b-1)}(P)$, and*

$$u_k = \left\lceil \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)u_{k-1}\right)\right\rceil \text{ for } k \geq b.$$

If $u_{b-1} \leq \binom{n}{2} - (n-1)(n-2b-1)$, then for every k such that $b-1 \leq k \leq (n-3)/2$,

$$E_{\leq k}(P) \geq u_k.$$

Proof. We proceed by induction on k. If $k = b-1$ the result holds by assumption. Let π be an arbitrary half-period of the circular sequence of P. Assume that $k \geq b$ and $E_{\leq k-1}(P) \geq u_{k-1}$. Let $s = s(k+1, \pi)$; by Lemma 10 (applied to $k+1$),

$$E_{\geq k+1}(P) \leq (n-2k-3)\, E_k(P) - \frac{s}{2}\left(E_k(P) - (n-1)\right).$$

If $s = 0$ or $E_k(P) \geq n-1$, then $E_{\geq k+1}(P) \leq (n-2k-3)E_k(P)$. Thus

$$\binom{n}{2} - E_{\leq k}(P) \leq (n-2k-3)\left(E_{\leq k}(P) - E_{\leq k-1}(P)\right),$$

and by induction

$$\begin{aligned} E_{\leq k}(P) &\geq \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)E_{\leq k-1}(P)\right) \\ &\geq \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)u_{k-1}\right), \end{aligned}$$

which implies that $E_{\leq k}(P) \geq u_k$ by the definition of u_k. Now assume $s > 0$ and $E_k(P) < n-1$. By Lemma 10,

$$\begin{aligned} E_{\geq k+1}(P) &\leq (n-2k-3)E_k(P) - \frac{s}{2}\left(E_k(P) - (n-1)\right) \\ &= (n-2k-3-\frac{s}{2})E_k(P) + \frac{s}{2}(n-1). \end{aligned}$$

Note that $s = s(k+1,\pi) \leq n-2k-3$ because at least one of the $n-2k-2$ elements of $C(k+1, \pi_0)$ must leave the middle $n-2k-2$ positions. Because $E_k(P) < n-1$, it follows that

$$E_{\geq k+1}(P) \leq (n-2k-3-\frac{s}{2})(n-1) + \frac{s}{2}(n-1) = (n-1)(n-2k-3),$$

and so

$$E_{\leq k}(P) = \binom{n}{2} - E_{\geq k+1}(P) \geq \binom{n}{2} - (n-1)(n-2k-3).$$

To finish the proof, we prove by induction that for $b-1 \leq k \leq (n-5)/2$,

$$\binom{n}{2} - (n-1)(n-2k-3) \geq u_k.$$

For the base case, note that by hypothesis

$$u_{b-1} \le \binom{n}{2} - (n-1)(n-2b-1)$$

Assume that $k \ge b$ and $u_{k-1} \le \binom{n}{2} - (n-1)(n-2k-1)$. By definition of u_k,

$$\begin{aligned} u_k &= \left\lceil \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)u_{k-1}\right)\right\rceil \\ &\le \left\lceil \frac{1}{n-2k-2}\left[\binom{n}{2} + (n-2k-3)\left(\binom{n}{2} - (n-1)(n-2k-1)\right)\right]\right\rceil \\ &= \left\lceil \binom{n}{2} - \frac{n-2k-1}{n-2k-2}(n-1)(n-2k-3)\right\rceil \le \binom{n}{2} - (n-1)(n-2k-3), \end{aligned}$$

which completes the induction. ◀

The next lemma serves to estimate u_k in the previous result.

▶ **Lemma 12.** *If u_k is defined recursively as in the previous lemma, then for $b-1 \le k \le (n-5)/2$,*

$$u_k \ge \binom{n}{2} - \sqrt{\frac{n-2k-2}{n-2b}}\left(\binom{n}{2} - u_{b-1}\right).$$

Proof. We prove this by induction on k. If $k = b-1$, then the right side of the inequality equals u_{b-1}. Assume that $k \ge b$ and

$$u_{k-1} \ge \binom{n}{2} - \sqrt{\frac{n-2k-4}{n-2b}}\left(\binom{n}{2} - u_{b-1}\right).$$

By induction and the definition of u_k,

$$\begin{aligned} u_k &\ge \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)u_{k-1}\right) \\ &\ge \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)\left[\binom{n}{2} - \sqrt{\frac{n-2k-4}{n-2b}}\left(\binom{n}{2} - u_{b-1}\right)\right]\right) \\ &= \binom{n}{2} - \frac{n-2k-3}{n-2k-2}\sqrt{\frac{n-2k-4}{n-2b}}\left(\binom{n}{2} - u_{b-1}\right) \\ &\ge \binom{n}{2} - \sqrt{\frac{n-2k-2}{n-2b}}\left(\binom{n}{2} - u_{b-1}\right). \end{aligned}$$

◀

We are now ready to present the best lower bound of $E_{\le k}(P)$ for k close to $n/2$. The new bound depends on an initial lower bound. We use the result of Lemma 9 as our initial bound u_{b-1}. We optimized the value b to obtain the best possible bound for m-fold symmetric sets, and it happens to be at $b = 1/2 - 1/(6m)$. This is now the best improvement for at-most-k edges for m-fold symmetric sets where k is close to $n/2$.

▶ **Lemma 13.** *Let $m \ge 3$ be odd. For any set P of n points with m-fold symmetry and for any $n/2 - n/(6m) \le k \le \lceil n/2 \rceil - 2$,*

$$\frac{E_{\le k}(P)}{n^2} \ge \frac{1}{2} - \frac{\sqrt{3m}(7m-1)}{9m(m+1)}\sqrt{1 - \frac{2k}{n}} + \Theta\left(\frac{1}{n}\right).$$

Proof. Let $b-1 = \lceil n(\frac{1}{2} - \frac{1}{6m})\rceil$. By Lemma 9, we know that

$$u_{b-1} := 4\left(\frac{m}{m+1}\right)\left(\binom{b+1}{2} - \binom{r+1}{2}\right) + mr + m\binom{b - \frac{m-1}{2}\cdot\frac{n}{m}+1}{2}$$

is a lower bound of $E_{\leq(b-1)}(P)$ (where $k+1 = r \bmod \lceil m/2\rceil$). Note that

$$\begin{aligned}\frac{u_{b-1}}{n^2} &= \frac{2m}{m+1}\left(\frac{1}{2}-\frac{1}{6m}\right)^2 + \frac{m}{2}\left(\frac{1}{2}-\frac{1}{6m}-\frac{(m-1)}{2m}\right)^2 + \Theta\left(\frac{1}{n}\right)\\ &= \frac{9m^2-5m+2}{18m(m+1)} + \Theta\left(\frac{1}{n}\right),\end{aligned}$$

and

$$\frac{1}{n^2}\left(\binom{n}{2} - (n-1)(n-2b-1)\right) = \frac{3m-2}{6m} + \Theta\left(\frac{1}{n}\right).$$

Because $(9m^2-5m+2)/(18m(m+1)) - (3m-2)/(6m) = -(4m-4)/(9m(m+1)) < 0$, it follows that $u_{b-1} \leq \binom{n}{2} - (n-1)(n-2b-1)$ for n large enough. Therefore, by Lemma 11 and Lemma 12, if $b-1 \leq k \leq (n-5)/2$, then

$$E_{\leq k}(P) \geq \binom{n}{2} - \sqrt{\frac{n-2k-2}{n-2b}}\left(\binom{n}{2} - u_{b-1}\right).$$

Finally, note that

$$\begin{aligned}\frac{E_{\leq k}(P)}{n^2} &\geq \frac{1}{2} - \sqrt{\frac{n}{n-2b}}\sqrt{1-\frac{2k}{n}}\left(\frac{1}{2} - \frac{9m^2-5m+2}{18m(m+1)}\right) + \Theta\left(\frac{1}{n}\right)\\ &= \frac{1}{2} - \sqrt{3m}\sqrt{1-\frac{2k}{n}}\left(\frac{7m-1}{9m(m+1)}\right) + \Theta\left(\frac{1}{n}\right)\\ &= \frac{1}{2} - \frac{\sqrt{3m}(7m-1)}{9m(m+1)}\sqrt{1-\frac{2k}{n}} + \Theta\left(\frac{1}{n}\right).\end{aligned}$$

◀

3 Even symmetry: exact crossing number

In this section, we prove Theorem 4. Throughout this section, m is an even integer. A construction achieving the stated crossing number is presented in Section 3.2. To prove the lower bound sym-$\overline{\text{cr}}_m(K_n) \geq \frac{1}{2}\binom{n}{4} + \frac{1}{2}\binom{n/2}{2} + \frac{1}{6}\binom{m/2-1}{2}n^2$, we use Theorem 1 followed by Lemma 7.

3.1 Lower bound

Let D be a rectilinear m-fold symmetric drawing of the complete graph K_n with set of vertices P. For $0 \leq k \leq n/2-2$, define r_k so that $k+1 = r_k \bmod (m/2)$. Then

$$\begin{aligned} cr(P) &= \sum_{k=0}^{n/2-2}(n-2k-3)E_{\leq k}(P) - \frac{3}{4}\binom{n}{3} + (1+(-1)^{n+1})\frac{1}{8}\binom{n}{2}\\ &\geq \sum_{k=0}^{n/2-2}(n-2k-3)\left(4\binom{k+2}{2} - 4\binom{r_k+1}{2} + mr_k\right) - \frac{3}{4}\binom{n}{3}\end{aligned}$$

We rewrite the previous expression by replacing each k by its value in terms of the integers $0 \le q_k < n/m$ and $0 \le r_k < m/2$, where $k+1 = \frac{m}{2}q_k + r_k$.

$$\begin{aligned} cr(P) &\ge \sum_{k=0}^{n/2-2} (n - mq_k - 2r_k - 1)\left(4\binom{\frac{m}{2}q_k + r_k + 1}{2} - 4\binom{r_k+1}{2} + mr_k\right) - \frac{3}{4}\binom{n}{3} \\ &= \sum_{q=0}^{\frac{n}{m}-1}\sum_{r=0}^{\frac{m}{2}-1} (n - mq - 2r - 1)\left(4\binom{\frac{m}{2}q + r + 1}{2} - 4\binom{r+1}{2} + mr\right) - \frac{3}{4}\binom{n}{3} \\ &= \frac{n}{48}(n^3 - 6n^2 + m^2n - 6mn + 22n - 12) = \frac{1}{2}\binom{n}{4} + \frac{1}{2}\binom{n/2}{2} + \frac{n^2}{6}\binom{m/2-1}{2}. \end{aligned}$$

3.2 Upper bound: a point set with low crossing number

For an even integer m, and an arbitrary positive integer J, we construct a set A_J with $n = mJ$ points that achieves the minimum number of at-most-k-edges for every k among m-fold symmetric sets with n points; and consequently it also has the minimum number of crossings among all m-fold symmetric sets with n points.

▶ **Lemma 14.** *Let $m > 0$ be even. For every positive integer J, there is an m-fold symmetric set A_J with mJ points such that*

1. $E_{\le k}(A_J) = 4(\binom{k+2}{2} - \binom{r+1}{2}) + mr$, *for every* $0 \le k \le mJ/2 - 2$, *where* $k + 1 = r \pmod{m/2}$.
2. $\mathrm{Cr}(A_J) = \frac{1}{2}\binom{mJ}{4} + \frac{1}{2}\binom{mJ/2}{2} + \frac{1}{6}\binom{m/2-1}{2}(mJ)^2$.

Here is the construction. Let $d = 2/\sin(\pi/(mJ))$. We define the m-fold symmetric set A_J as follows (see Figure 6):

$$A_J = \left\{\left(d^j \cos\left(\frac{\pi(2aJ - j)}{mJ}\right), d^j \sin\left(\frac{\pi(2aJ - j)}{mJ}\right)\right) : 0 \le a \le m-1, 0 \le j \le J-1\right\}.$$

For every $0 \le a \le m-1$ and $0 \le j \le J-1$ we use the notation

$$p(j,a) = \left(d^j \cos\left(\frac{\pi(2aJ - j)}{mJ}\right), d^j \sin\left(\frac{\pi(2aJ - j)}{mJ}\right)\right).$$

▶ **Lemma 15.** *If $0 \le j_1 \le j_2 < J-1$, $0 \le a_1 \le a_2 \le m-1$, and $(j_1, a_1) \ne (j_2, a_2)$, then the line through $p(j_1, a_1)$ and $p(j_2, a_2)$ is a halving line of the outermost m-gon in A_n; that is, it separates the set $\{p(J-1, a) : 0 \le a \le m-1\}$ into two parts each with $m/2$ points.*

Proof. By rotational symmetry, assume that $a_1 = 0$ and for simplicity let $a_2 = a$. Furthermore, by using a dilation by a factor of d^{-j_1}, we can assume that $j_1 = 0$. Again by simplicity, we let $j_2 = j$, and we note that now we need to prove that the line through $p(0,0)$ and $p(j,a)$ halves the $c = J - j_1$ outer m-gon; that is, it halves the set $\{p(c,b) : 0 \le b \le m-1\}$. Note that $0 \le j \le c-1$, $0 \le a \le m-1$, and $(j,a) \ne (0,0)$. We will prove this assertion by showing that the points $p(c,b)$ and $p(c, b+m/2)$ lie on different sides of the line. To do this, we will show that the determinants D and D' that follow have different signs.

$$D = \begin{vmatrix} p(0,0) & 1 \\ p(j,a) & 1 \\ p(c,b) & 1 \end{vmatrix} \quad \text{and} \quad D' = \begin{vmatrix} p(0,0) & 1 \\ p(j,a) & 1 \\ p(c,b+m/2) & 1 \end{vmatrix}.$$

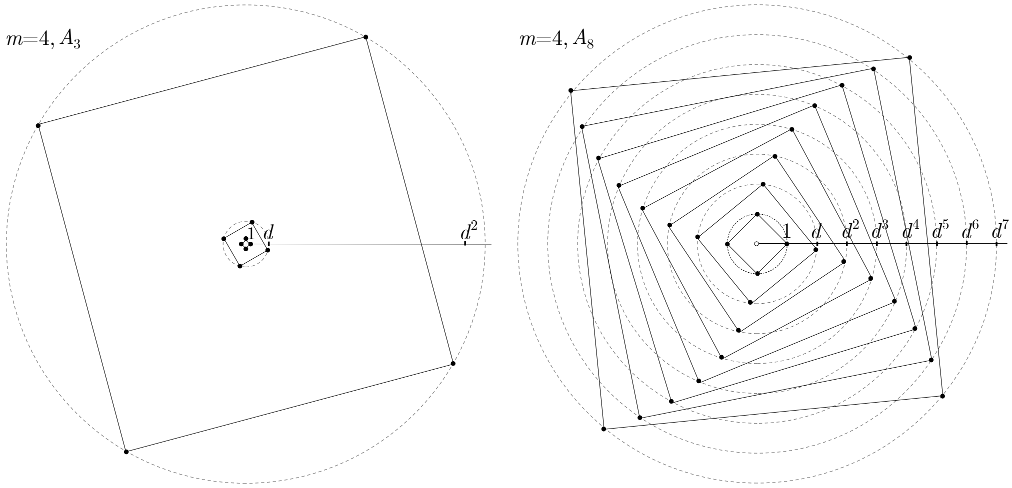

Figure 6 The 4-fold symmetric ($m = 4$) sets A_3 and A_8 described in the proof of Lemma 14. Since any more than 3 layers would be virtually impossible to see, the set A_8 is shown in logarithmic-polar scale to appreciate the rotations from one layer to the next.

A direct calculation shows that

$$D = \begin{vmatrix} p(0,0) & 1 \\ p(j,a) & 1 \\ p(c,b) & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ d^j \cos(\pi(2aJ - j)/mJ) & d^j \sin(\pi(2aJ - j)/mJ) & 1 \\ d^c \cos(\pi(2bJ - c)/mJ) & d^c \sin(\pi(2bJ - c)/mJ) & 1 \end{vmatrix}$$

$$= d^j \sin\left(\frac{\pi(2aJ - j)}{mJ}\right) - d^c \left[\sin\left(\frac{\pi(2bJ - c)}{mJ}\right) + d^j \sin\left(\frac{\pi(2aJ - 2bJ + c - j)}{mJ}\right)\right],$$

and

$$D' = \begin{vmatrix} p(0,0) & 1 \\ p(j,a) & 1 \\ p(c,b+m/2) & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ d^j \cos(\pi(2aJ - j)/mJ) & d^j \sin(\pi(2aJ - j)/mJ) & 1 \\ -d^c \cos(\pi(2bJ - c)/mJ) & -d^c \sin(\pi(2bJ - c)/mJ) & 1 \end{vmatrix}$$

$$= d^j \sin\left(\frac{\pi(2aJ - j)}{mJ}\right) + d^c \left[\sin\left(\frac{\pi(2bJ - c)}{mJ}\right) + d^j \sin\left(\frac{\pi(2aJ - 2bJ + c - j)}{mJ}\right)\right].$$

We first deal with the case where $j \geq 1$. Because $1 \leq c - j \leq J - 1$, it follows that $c - j \not\equiv 0 \pmod{J}$. Hence $2aJ - 2bJ + c - j \not\equiv 0 \pmod{mJ}$, and thus

$$\left|\sin\left(\frac{\pi(2aJ - 2bJ + c - j)}{mJ}\right)\right| \geq \sin\left(\frac{\pi}{mJ}\right).$$

It follows that

$$\begin{aligned} & d^c \left|\sin\left(\frac{\pi(2bJ - c)}{mJ}\right) + d^j \sin\left(\frac{\pi(2aJ - 2bJ + c - j)}{mJ}\right)\right| \\ & \geq d^c \left(d^j \left|\sin\left(\frac{\pi(2aJ - 2bJ + c - j)}{mJ}\right)\right| - \left|\sin\left(\frac{\pi(2bJ - c)}{mJ}\right)\right|\right) \\ & \geq d^c \left(d^j \sin\left(\frac{\pi}{mJ}\right) - 1\right) = d^c(2d^{j-1} - 1) \geq d^c(2 - 1) = d^c \\ & \qquad \geq d^{j+1} > d^j \geq d^j \left|\sin\left(\frac{\pi(2aJ - j)}{mJ}\right)\right|. \end{aligned}$$

Now assume $j = 0$. Note that $c \geq 1$ and $1 \leq a \leq m-1$. Because $1 \leq c \leq J-1$, it follows that $aJ - 2bJ + c \not\equiv 0 \pmod{mJ}$, and thus

$$\left|\cos\left(\frac{\pi(aJ-2bJ+c)}{mJ}\right)\right| \geq \cos\left(\frac{\pi}{2} - \frac{\pi}{mJ}\right) = \sin\left(\frac{\pi}{mJ}\right).$$

Thus

$$\begin{aligned}
&d^c\left|\sin\left(\frac{\pi(2bJ-c)}{mJ}\right) + d^j\sin\left(\frac{\pi(2aJ-2bJ+c-j)}{mJ}\right)\right| \\
&= d^c\left|\sin\left(\frac{\pi(2bJ-c)}{mJ}\right) + \sin\left(\frac{\pi(2aJ-2bJ+c)}{mJ}\right)\right| \\
&= d^c\left|\sin\left(\frac{\pi a}{m}\right)\cos\left(\frac{\pi(aJ-2bJ+c)}{mJ}\right)\right| \\
&\geq d\left|\sin\left(\frac{\pi a}{m}\right)\right|\sin\left(\frac{\pi}{mJ}\right) = 2\left|\sin\left(\frac{\pi a}{m}\right)\right| \\
&\qquad > 2\left|\sin\left(\frac{\pi a}{m}\right)\cos\left(\frac{\pi a}{m}\right)\right| = \left|\sin\left(\frac{2\pi a}{m}\right)\right| = d^j\left|\sin\left(\frac{\pi(2aJ-j)}{mJ}\right)\right|
\end{aligned}$$

In either case, the previous inequalities show that the product DD' is negative, and thus $p(c,b)$ and $p(c,b+m/2)$ lie on different sides of the line through $p(0,0)$ and $p(j,a)$. ◀

▶ **Lemma 16.** *For every $1 \leq a \leq m/2-1$ the edge $p(J-1,0)p(J-1,a)$ is an $(a-1)$-edge of A_J.*

Proof. Suppose that $1 \leq a \leq m/2-1$. It is enough to show that the ray $p(J-1,0)p(J-1,a)$ leaves the set of points $\{p(J-1,b) : 1 \leq b \leq a-1\}$ on its right side, and the rest of points of A_J on its left side. Because the set $\{p(J-1,b) : 0 \leq b \leq J-1\}$ is a regular m-gon, then among these points only those with $1 \leq b \leq a-1$ are on the right side of $p(J-1,0)p(J-1,a)$. To finish the proof, we show that the rest of A_J, namely the set $A'_J = \{p(j,b) : 0 \leq j \leq J-2, 0 \leq b \leq m-1\}$ lies on the left side of $p(J-1,0)p(J-1,a)$. If $J = 1$, this is vacuously true, otherwise if R denotes the distance from the origin to the point $p(J-1,0)$, then the distance from the origin to the segment $p(J-1,0)p(J-1,a)$ is $R\cos(\pi a/m)$, and by construction A'_J is inside a circle of radius $R/d = R\sin(\pi/(mJ))/2$ centered at the origin. Finally,

$$\begin{aligned}
\cos\left(\frac{\pi a}{m}\right) &\geq \cos\left(\frac{\pi(m/2-1)}{m}\right) = \sin\left(\frac{\pi}{m}\right) = 2\sin\left(\frac{\pi}{2m}\right)\cos\left(\frac{\pi}{2m}\right) \\
&\geq 2\sin\left(\frac{\pi}{mJ}\right)\cos\left(\frac{\pi}{4}\right) > 2\sin\left(\frac{\pi}{mJ}\right)\cdot\frac{1}{4} = \frac{1}{2}\sin\left(\frac{\pi}{mJ}\right),
\end{aligned}$$

thus A'_J lies on the left side of the ray $p(J-1,0)p(J-1,a)$. ◀

Proof of Lemma 14. We proceed to find an upper bound for the number of at-most-k-edges of A_J. By rotational symmetry consider the at-most-k-edges of the form $p(j,0)p(i,a)$ with $0 \leq i \leq j \leq n-1$, $0 \leq a \leq m-1$, and $(j,0) \neq (i,a)$.

First suppose that $k \leq m/2-2$. If $j = J-1$, then according to Lemma 16 the edges from $p(J-1,0)$ to $p(i,a)$ with $i = J-1$ and $1 \leq a \leq k+1$ are all at-most-k-edges. Moreover, if $i < J-1$ and a is arbitrary, then the line $p(J-1,0)p(i,a)$ separates the sets of points $\{p(J-1,b) : 1 \leq b \leq m/2-1\}$ and $\{p(J-1,b) : m/2+1 \leq b \leq m-1\}$. Thus the edge $p(J-1,0)p(i,a)$ is a k-edge with $k \geq m/2-1$. Finally, if $j < J-1$ and $i < J-1$, then by Lemma 15 the edge $p(J-1,0)p(i,a)$ is a k-edge with $k \geq m/2$. Hence in this case all of the at-most-k-edges have both vertices in the outer m-gon where $i = j = J - 1$, and there are exactly $2(k+1)$ such edges of the form $p(j,0)p(i,a)$. Hence there are exactly $m(k+1)$ at-most-k-edges in A_J. In other words, if $k \le m/2 - 2$, then

$$E_{\le k}(A_J) = m(k+1). \tag{3}$$

Second, suppose that $m/2 - 1 \le k \le mJ/2 - 3$. By Lemma 16 all of the edges with vertices in the outer m-gon, except the main diagonals, are at-most-$(m/2-1)$-edges, and so they are all at-most-k-edges as well. There are exactly $\binom{m}{2} - m/2$ such edges. In addition, by considering a radial sweep from each of the convex hull vertices $p(J-1,a)$ of A_J, we see that each of them is the endpoint of exactly two j-edges for every $m/2 \le j \le k$, where the other endpoint is not a vertex of the outer m-gon. There are $2m(k - (m/2-1))$ such edges. Finally, by Lemma 15, every j-edge of A'_J is a $(j + m/2)$-edge of A_J. Hence if $m/2 - 1 \le k \le mJ/2 - 3$, then

$$\begin{aligned} E_{\le k}(A_J) &= E_{\le (k-m/2)}(A'_J) + \binom{m}{2} - \frac{m}{2} + 2m\left(k - \left(\frac{m}{2} - 1\right)\right) \\ &= E_{\le (k-m/2)}(A'_J) + 2m(k+1) - m\left(\frac{m}{2} - 1\right). \end{aligned} \tag{4}$$

From (3) and (4), we conclude that

$$E_{\le k}(A_J) = E_{\le k-m/2}(A'_J) + 2m(k+1) - m \cdot \min(k+1, m/2-1).$$

Therefore the set A_J achieves equality in Lemma 6. Following the proof of Lemma 7 we conclude that A_J achieves equality throughout as well. This assertion proves the first part, and the second follows because again the set A_J achieves equality every time Lemma 7 is used in the proof of Theorem 4. ◀

4 Odd symmetry: bounds on the crossing number

In this section, we prove the bounds stated in Theorem 5 on the rectilinear crossing number of configurations with odd symmetry. We start with the lower bound, Inequality 5, in Section 4.1; and then present a construction with few crossings that takes care of the upper bound, Inequality 6, in Section 4.2. Throughout this section, we assume that m and n are positive integers such that $m \ge 3$ is odd and $m | n$.

4.1 Lower bound

We are now ready to present our lower bound in Theorem 5 on the crossing number for odd symmetry, namely,

$$\text{sym-}\overline{\text{cr}}_m(K_n) \ge \left(\frac{1}{2} - \frac{m-1}{2m^2} - \frac{97m - 31}{270m^3(m+1)}\right)\binom{n}{4} + \Theta(n^3). \tag{5}$$

This bound follows from Theorem 1 and Lemmas 9 and 13. The use of Lemma 13 improves the weaker bound obtained by using only Lemma 9, that is,

$$\text{sym-}\overline{\text{cr}}_m(K_n) \ge \left(\frac{1}{2} - \frac{m-1}{2m^2} - \frac{3m - 1}{8m^3(m+1)}\right) + \Theta(n^3).$$

(For reference, $3/8 = 0.375$ and $97/270 = 0.3\overline{592}$.)

Proof of Inequality 5. Let $m \geq 3$ be odd an P be an m-fold symmetric set of n points in general position in the plane. By Theorem 1, we have

$$\begin{aligned}\operatorname{Cr}(P) &= \sum_{k=0}^{\lfloor n/2\rfloor-2}(n-2k-3)E_{\leq k}(P)+\Theta(n^3)\\ &= \binom{n}{4}\cdot 24\sum_{k=0}^{\lfloor n/2\rfloor-2}\left(1-\frac{2k}{n}\right)\frac{E_{\leq k}(P)}{n^2}\cdot\frac{1}{n}+\Theta(n^3).\end{aligned}$$

Now we bound $E_{\leq k}(P)$ using Lemma 9 in the range $0 \leq k/n \leq 1/2-1/(6m)$ and using Lemma 13 when $1/2-1/(6m) \leq k/n \leq 1/2$. For $0 \leq k/n \leq 1/2-1/(6m)$, we have that

$$\begin{aligned}\frac{E_{\leq k}(P)}{n^2} &\geq \frac{4m}{m+1}\cdot\frac{k^2}{2n^2}+\frac{m}{2n^2}\cdot\max\left(0,\left(k-\frac{(m-1)n}{2m}\right)^2\right)+\Theta\left(\frac{1}{n}\right)\\ &= \frac{2m}{m+1}\left(\frac{k}{n}\right)^2+\frac{m}{2}\left(\max\left(0,\frac{k}{n}-\frac{m-1}{2m}\right)\right)^2+\Theta\left(\frac{1}{n}\right),\end{aligned}$$

and for $1/2-1/(6m) < k \leq 1/2$,

$$\frac{E_{\leq k}(P)}{n^2} \geq \frac{1}{2}-\frac{\sqrt{3m}(7m-1)}{9m(m+1)}\sqrt{1-\frac{2k}{n}}+\Theta\left(\frac{1}{n}\right).$$

Therefore,

$$\begin{aligned}\operatorname{Cr}(P) \geq{}& \binom{n}{4}\int_0^{1/2-1/(6m)} 24\,(1-2x)\left(\frac{2m}{m+1}x^2+\frac{m}{2}\left(\max\left(0,x-\frac{m-1}{2m}\right)\right)^2\right)dx\\ &+\binom{n}{4}\int_{1/2-1/(6m)}^{1/2} 24(1-2x)\left(\frac{1}{2}-\frac{\sqrt{3m}(7m-1)}{9m(m+1)}\sqrt{1-2x}\right)dx+\Theta(n^3)\\ ={}& \binom{n}{4}\left(\frac{9m^3-9m^2+3m+1}{18m^3}+\frac{135m^4+38m+31}{270m^3(m+1)}\right)+\Theta(n^3)\\ ={}& \left(\frac{1}{2}-\frac{m-1}{2m^2}-\frac{97m-31}{270m^3(m+1)}\right)\binom{n}{4}+\Theta(n^3).\end{aligned}$$

◀

4.2 Upper bound: a point set with low crossing number

In this section, we provide a set of points with few crossings for odd symmetry $m \geq 5$, to prove the upper bound in Theorem 5, namely,

$$\operatorname{sym-}\overline{\operatorname{cr}}_m(K_n) \leq \left(\frac{1}{2}-\frac{m-1}{2m^2}\right)\binom{n}{4}+\Theta(n^3). \tag{6}$$

This construction is similar to the case where m is even. It consists of a wing set W of J points almost on a line, whose x-coordinates increase exponentially, and $m-1$ copies of W obtained by a $(2\pi a/m)$-rotation for $1 \leq a \leq m-1$ so that the orbit of each point in W is a regular m-gon, see Figure 7. The difference is that in the even case the set W can be in convex position, but in the odd case it matters whether W has fewer crossings. In Section 4.2.1, we calculate the number of crossings in our construction with an arbitrary wing set W, and in Section 4.2.2, we present a suitable wing W with few crossings.

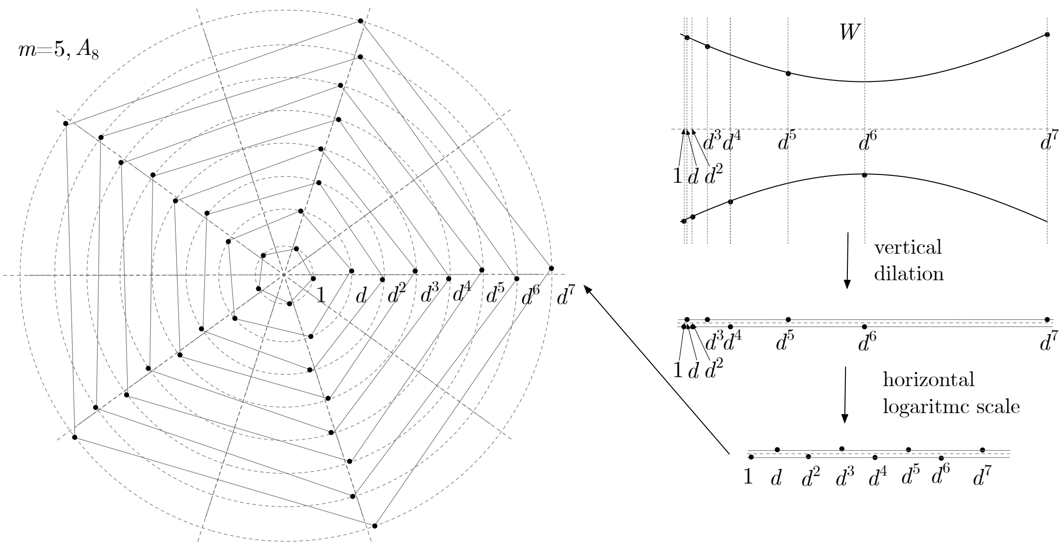

Figure 7 A 5-fold symmetric set of 40 points generated by the $(5, 8)$-wing set W.

4.2.1 Construction using an arbitrary wing set

We start by defining (m, J)-wing sets. Let $W = \{w_1, w_2, \ldots, w_J\}$ be a set of J points. For every $0 \leq a \leq m-1$, let $W_a := \mathcal{R}^a(W)$ be the rotation of W by $2\pi a/m$. We call each of the sets W_a a *wing*, and for $1 \leq j \leq J$ we refer to the orbit $L_j := [w_j]$ (which is a regular m-gon with vertices w_j and its $m-1$ rotations) as a *layer*.

As in the even case, let $d = 2/\sin(\pi/mJ)$. We say that W is a (m, J)-*wing* if there is $\varepsilon > 0$ such that the following conditions hold:

1. Every w_j is in a ε-neighborhood of the point with coordinates $(d^{j-1}, 0)$.
2. The line by any two points in W separates all the points in $W_1 \cup W_2 \cup \cdots \cup W_{(m-1)/2}$ from $W_{(m+1)/2} \cup \cdots \cup W_{m-1}$.
3. For every $2 \leq j \leq J$, the set $L_1 \cup L_2 \cup \cdots \cup L_{j-1}$ is in the center of the star formed by the larger diagonals of L_j.

▶ **Lemma 17.** *Let $m \geq 5$ be odd and J be a positive integer. For every (m, J)- wing set W, if $A_J = \cup_{a=0}^{m-1} W_a$, then A_J has mJ points and*

$$\mathrm{Cr}(A_J) = \begin{cases} mCr(W) + \frac{1}{48}m(m-1)J[J^3(m^2+1) \\ \quad -2J^2(2m+3) + J(m^2-7m+20) + m - 3] & \text{if } m \equiv 1 \pmod 4, \\ mCr(W) + \frac{1}{48}mJ[J^3(m^3-m^2+m-1) \\ \quad +J^2(-4m^2-6m+6) + J(m^3-8m^2+27m-8) + m^2 - 9] \\ & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Proof. For every $1 \leq j \leq J$, let $A_j = L_1 \cup L_2 \cup \cdots \cup L_j$. We count the number of crossings in A_J by determining its vector of k-edges; that is by calculating

$$\overrightarrow{E}(A_J) := (E_0(A_J), E_1(A_J), \ldots, E_{\lfloor mJ/2 \rfloor - 1}).$$

This is enough because according to Theorem 1,

$$\mathrm{Cr}(A_J) = 3\binom{mJ}{4} - \overrightarrow{E}(A_J) \cdot \overrightarrow{v} \tag{7}$$

where $\overrightarrow{v}$ is the $\lfloor mJ/2\rfloor$-vector whose k^{th} entry is $(k-1)(mJ-1-k)$.

We determine $\overrightarrow{E}(A_J)$ by considering three different types of k-edges: edges determined by two points (1) in the same layer, (2) in the same wing, and (3) in different layer and different wing. We partition the k-edge vector into three corresponding parts, denoting them by $\overrightarrow{E}_{\text{layer}}(A_J)$, $\overrightarrow{E}_{\text{wing}}(A_J)$, and $\overrightarrow{E}_{\text{diff}}(A_J)$, respectively. Clearly,

$$\overrightarrow{E}(A_J) = \overrightarrow{E}_{\text{layer}}(A_J) + \overrightarrow{E}_{\text{wing}}(A_J) + \overrightarrow{E}_{\text{diff}}(A_J). \tag{8}$$

We write each of these vectors in terms of $\overrightarrow{E}(W) = (e_0, e_1, \ldots, e_{\lfloor J/2\rfloor -1})$. By symmetry and the first property,

$$\overrightarrow{E}_{\text{wing}}(A_J) = m(\underbrace{0,0,\ldots,0}_{\frac{m-1}{2}\cdot J}, e_0, e_1, \ldots, e_{\lfloor J/2\rfloor -1}).$$

It follows that

$$\begin{aligned}
\overrightarrow{E}_{\text{wing}}(A_J)\cdot\overrightarrow{v} &= m\sum_{k=(m-1)J/2}^{\lfloor mJ/2\rfloor -1} k(mJ-k-2)e_{k-(m-1)J/2}\\
&= m\sum_{t=0}^{\lfloor J/2\rfloor -1}(t+\tfrac{1}{2}(m-1)J)(mJ-2-t-\tfrac{1}{2}(m-1)J)e_t\\
&= m\left(\sum_{t=0}^{\lfloor J/2\rfloor -1} t(J-2-t)e_t\right) + \frac{1}{4}m(m-1)J(mJ+J-4)\sum_{t=0}^{\lfloor J/2\rfloor -1} e_t.
\end{aligned}$$

Clearly $\sum_{t=0}^{\lfloor J/2\rfloor -1} e_t = \binom{J}{2}$ and by Theorem 1, $Cr(W) = 3\binom{J}{4} - \sum_{t=0}^{\lfloor J/2\rfloor -1} t(J-2-t)e_t$. Thus

$$\overrightarrow{E}_{\text{wing}}(A_J)\cdot\overrightarrow{v} = 3m\binom{J}{4} - mCr(W) + \frac{1}{8}m(m-1)J(mJ+J-4)J(J-1). \tag{9}$$

We now look at $\overrightarrow{E}_{\text{diff}}(A_J)$. Consider a point $p_2 \in L_{j_2}\cap W_a$ for some $1 \le j_2 \le J$ and $0 \le a \le m-1$. Note that $p_2 \in \partial A_{j_2}$ and thus, there are exactly two points in $A_{j_2-1}\setminus W_a$ forming a k-edge of A_{j_2} with p for any $\frac{1}{2}(m-1) \le k \le \frac{1}{2}(m-1)j_2$. Suppose that p_1p_2 is a k-edge of A_{j_2} with $p_1 \in L_{j_1}$ for some $1 \le j_1 < j_2$ and $p_1 \notin W_a$. Then p_1p_2 divides the plane into two open halfplanes H^- and H^+ with k and $mJ-2-k$ points of A_{j_2}, respectively. Note that by construction H^- contains exactly $(m-1)/2$ points of L_{j_2} and $(m+1)/2$ points of each of the layers $L_{j_2+1}, L_{j_2+2}, \ldots, L_J$. Therefore, p_1p_2 leaves $k_1 := k+(J-j_2)\frac{1}{2}(m+1)$ points of A_J on one side and $k_2 := mJ-2-k-(J-j_2)\frac{1}{2}(m+1)$ on the other. So, if $k_3 = \min(k_1,k_2)$, then p_1p_2 is a k_3-edge of A_J. Even when we do not know whether $k_3 = k_1$ or $k_3 = k_2$, its contribution to $\overrightarrow{E}_{\text{diff}}(A_J)\cdot\overrightarrow{v}$ is $k_3(mJ-2-k_3) = k_1(mJ-2-k_1) = k_2(mJ-2-k_2)$ because $k_1+k_2 = mJ-2$. For this reason, we extend the $\lfloor mJ/2\rfloor$-vector $\overrightarrow{E}_{\text{diff}}(A_J)$ to the $(mJ-1)$-vector $\overrightarrow{E}^*_{\text{diff}}(A_J)$ that counts a p_1p_2 edge as above in the k_1 entry. Thus $\overrightarrow{E}_{\text{diff}}(A_J)\cdot\overrightarrow{v} = \overrightarrow{E}^*_{\text{diff}}(A_J)\cdot\overrightarrow{v}^*$, where the $(mJ-1)$-vector $\overrightarrow{v}^*$ satisfies that its k^{th} and $(mJ-1+k)^{th}$ entries are $(k-1)(mJ-1-k)$.

Therefore, for every $2 \le j_2 \le J$, the contribution to $\overrightarrow{E}^*_{\text{diff}}(A_J)$ of k-edges from layer L_{j_2} to A_{j_2-1} is exactly

$$(\underbrace{0,0,\ldots,0}_{\frac{1}{2}(m-1)}, \underbrace{\underbrace{0,0,\ldots,0}_{\frac{1}{2}(m+1)},\ldots,\underbrace{0,0,\ldots,0}_{\frac{1}{2}(m+1)}}_{J-j_2}, \underbrace{2m,2m,\ldots,2m}_{\frac{1}{2}(m-1)(j_2-1)}, \underbrace{0,0,\ldots,0}_{\frac{1}{2}(m-1)J+(j_2-1)}).$$

Thus, adding over all $2 \le j_2 \le J$, we have

$$\begin{aligned}\overrightarrow{E}_{\text{diff}}(A_J) \cdot \overrightarrow{v} &= \overrightarrow{E}^*_{\text{diff}}(A_J) \cdot \overrightarrow{v}^* \\ &= \sum_{j_2=2}^{J} \sum_{k=\frac{1}{2}(m-1)+(J-j_2)\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)+(J-j_2)\frac{1}{2}(m-1)+(j_2-1)\frac{1}{2}(m-1)-1} 2mk(mJ-k-2) \\ &= \frac{1}{48} m(m-1)J(J-1)(5J^2m^2 + Jm^2 - 26Jm + 2m - J^2 + J + 26). \end{aligned} \tag{10}$$

Finally, we look at $\overrightarrow{E}_{\text{layer}}(A_J)$. Suppose that p_1p_2 is a k-edge of L_j for some $1 \le j \le J$. Let H be the open halfplane determined by p_1p_2 that contains exactly k points of L_j. Note that for each $0 \le k \le \frac{1}{2}(m-3)$ there are exactly m such edges. We say that p_1p_2 is an *even edge* if k is even and an *odd edge* if k is odd. We denote the contributions of the even an odd edges to $\overrightarrow{E}_{\text{layer}}(A_J)$ by $\overrightarrow{E}_{\text{even}}(A_J)$ and $\overrightarrow{E}_{\text{odd}}(A_J)$, respectively. As before, we extend each of these $\lfloor mJ/2 \rfloor$-vectors to $(mJ-1)$-vectors so that

$$\overrightarrow{E}_{\text{layer}}(A_J) \cdot \overrightarrow{v} = \overrightarrow{E}^*_{\text{layer}}(A_J) \cdot \overrightarrow{v}^* = \overrightarrow{E}^*_{\text{even}}(A_J) \cdot \overrightarrow{v}^* + \overrightarrow{E}^*_{\text{odd}}(A_J) \cdot \overrightarrow{v}^*, \tag{11}$$

and consider two cases depending on $m \pmod 4$.

First assume that $m \equiv 1 \pmod 4$. If k is even, then p_1p_2 leaves exactly $(m-1)/2$ points of each $L_{j+1}, L_{j+2}, \cdots, L_J$ in H^-. That is, p_1p_2 leaves $k + (J-j)\frac{1}{2}(m-1)$ points of A_J on H. Therefore, for every $1 \le j \le J$, the contribution of even edges of L_j to $\overrightarrow{E}^*_{\text{even}}(A_J)$ is exactly

$$(\underbrace{0,0,\ldots,0}_{\frac{1}{2}(m-1)(J-j)}, \underbrace{m,0,m,0,\ldots,m,0}_{\frac{1}{2}(m-1)}, \underbrace{0,0,\ldots,0}_{\frac{1}{2}(m-1)(J+j-1)+J-1}).$$

Adding these vectors over all $1 \le j \le J$, we obtain

$$\overrightarrow{E}^*_{\text{even}}(A_J) = (\underbrace{m,0,m,0,\ldots,m,0}_{\frac{1}{2}(m-1)J}, \underbrace{0,0,\ldots,0}_{\frac{1}{2}(m+1)J-1}).$$

Similarly, if k is odd, then p_1p_2 leaves exactly $(m+1)/2$ points of each $L_{j+1}, L_{j+2}, \cdots, L_J$ on H. Thus p_1p_2 leaves $k + (J-j)\frac{1}{2}(m+1)$ points of A_J on H. Therefore, for every $1 \le j \le J$, the contribution of odd edges of L_j to $\overrightarrow{E}^*_{\text{odd}}(A_J)$ is exactly

$$(\underbrace{0,0,\ldots,0}_{\frac{1}{2}(m+1)(J-j)}, \underbrace{0,m,0,m,\ldots,0,m}_{\frac{1}{2}(m-1)}, \underbrace{0,0,\ldots,0}_{\frac{1}{2}(m-1)(J+j-1)-J}).$$

Adding these vectors over all $1 \le j \le J$, we obtain

$$\overrightarrow{E}^*_{\text{odd}}(A_J) = (\underbrace{\underbrace{0,m,0,m,\ldots,m,0}_{\frac{1}{2}(m+1)}, \underbrace{0,m,0,m,\ldots,m,0}_{\frac{1}{2}(m+1)}, \ldots, \underbrace{0,m,0,m,\ldots,m,0}_{\frac{1}{2}(m+1)}}_{J}, \underbrace{0,0,\ldots,0}_{\frac{1}{2}(m-1)J-1}).$$

Now assume that $m \equiv 3 \pmod 4$. If k is even, then p_1p_2 leaves exactly $(m+1)/2$ points of each $L_{j+1}, L_{j+2}, \cdots, L_J$ on H. That is, p_1p_2 leaves $k + (J-j)\frac{1}{2}(m+1)$ points of A_J on

H. Therefore, for every $1 \leq j \leq J$, the contribution of even edges of L_j to $\overrightarrow{E}^*_{\text{even}}(A_J)$ is exactly

$$(\underbrace{0,0,\ldots,0}_{\frac{1}{2}(m+1)(J-j)},\underbrace{m,0,m,0,\ldots,m}_{\frac{1}{2}(m-1)},\underbrace{0,0,\ldots,0}_{\frac{1}{2}(m+1)(J+j)-J}).$$

Adding these vectors over all $1 \leq j \leq J$, we obtain

$$\overrightarrow{E}^*_{\text{even}}(A_J) = (\underbrace{m,0,m,0,\ldots,m,0}_{\frac{1}{2}(m+1)J},\underbrace{0,0,\ldots,0}_{\frac{1}{2}(m-1)J-1}).$$

Similarly, if k is odd, then p_1p_2 leaves exactly $(m-1)/2$ points of each $L_{j+1}, L_{j+2}, \cdots, L_J$ on H. Thus p_1p_2 leaves $k + (J-j)\frac{1}{2}(m-1)$ points of A_J on H. Therefore, for every $1 \leq j \leq J$, the contribution of odd edges of L_j to $\overrightarrow{E}^*_{\text{odd}}(A_J)$ is exactly

$$(\underbrace{0,0,\ldots,0}_{\frac{1}{2}(m-1)(J-j)},\underbrace{0,m,0,m,\ldots,m,0}_{\frac{1}{2}(m-1)},\underbrace{0,0,\ldots,0}_{\frac{1}{2}(m-1)(J+j-1)+J-1})$$

Adding these vectors over all $1 \leq j \leq J$, we obtain

$$\overrightarrow{E}^*_{\text{odd}}(A_J) = (\underbrace{\underbrace{0,m,0,m,\ldots,m,0}_{\frac{1}{2}(m-1)},\underbrace{0,m,0,m,\ldots,m,0}_{\frac{1}{2}(m-1)},\ldots,\underbrace{0,m,0,m,\ldots,m,0}_{\frac{1}{2}(m-1)}}_{J},\underbrace{0,0,\ldots,0}_{\frac{1}{2}(m+1)J-1}).$$

Therefore,

$$\overrightarrow{E}^*_{\text{even}}(A_J)\cdot\overrightarrow{v*} = \begin{cases} \frac{1}{48}m(m-1)J(mJ-J-4)(2mJ+J-4) & \text{if } m \equiv 1 \pmod 4 \\ \frac{1}{48}m(m+1)J(mJ+J-4)(2mJ-J-4) & \text{if } m \equiv 3 \pmod 4, \end{cases} \tag{12}$$

and

$$\overrightarrow{E}^*_{\text{odd}}(A_J)\cdot\overrightarrow{v*} = \begin{cases} \frac{1}{48}m(m-1)J[J^2(2m-1)(m+1)-3J(3m+1)+m+13] \\ \qquad\qquad \text{if } m \equiv 1 \pmod 4, \\ \frac{1}{48}m(m-3)J[J^2(2m+1)(m-1)-3J(3m-1)+m+11] \\ \qquad\qquad \text{if } m \equiv 3 \pmod 4. \end{cases} \tag{13}$$

The value of $\text{Cr}(A_J)$ is obtained from (7)-(13). ◀

4.2.2 A suitable wing set: proof of inequality 6

In contrast to the even case, the number of crossings in A_J is smaller when the set W has fewer crossings. It is tempting to use a set W achieving the rectilinear crossing number of K_J (or at least one of the best known sets with few crossings). However, it is not clear that these sets can be modified to satisfy the exponential increase on their x-coordinates. Instead, we use the so-called *double chain construction*, which consists of (almost) half of the points in each of two underlying curves, one convex and one concave, such that any tangent line to any of the curves separates them. This construction can consist of a convex curve above the x-axis and its symmetric about the x-axis each intersecting the ε-neighborhood of the point with coordinates $(d^{j-1}, 0)$ for every $1 \le j \le J$. The set W consists of one point in each of these ε-neighborhoods so that $\lfloor J/2 \rfloor$ points are on one of the curves and $\lceil J/2 \rceil$ on the other (see Figure 7). The set of points in each curve is called a *chain*. Any four points in the same chain generate a crossing, as well as any quadruple with one pair on one chain and one pair on the other. Thus $\mathrm{Cr}(W) = 2\binom{J/2}{4} + \binom{J/2}{2}^2$. It follows that, regardless of $m \pmod 4$,

$$\mathrm{Cr}(A_J) = \frac{1}{48} J^4 m^2 (m^2 - m + 1) + \Theta(J^3).$$

Setting $n = mJ$ (and since $\text{sym-}\overline{\mathrm{cr}}_m(K_n) \le \mathrm{Cr}(A_J)$), we obtain Inequality 6, which is the best upper bound we know for odd $m \ge 5$.

5 Conclusions and open questions

We study the rectilinear crossing number of complete graphs, $\text{sym-}\overline{\mathrm{cr}}_m(K_n)$, when the vertex set is constrained to have rotational m-fold symmetry. As our main contribution, we determine the exact value of this crossing number for even symmetry and provide lower and upper bounds for odd symmetry. We first provide a lower bound that covers all values of m. It results from a lower estimate on the number of at-most-k edges for m-fold symmetric configurations; via the crossing number identity, this estimate directly translates into a lower bound on $\text{sym-}\overline{\mathrm{cr}}_m(K_n)$. For even m, we determine the exact value of $\text{sym-}\overline{\mathrm{cr}}_m(K_n)$ by matching this lower bound with an explicit construction. For odd m, we sharpen the lower bound in the mid- and near-central ranges of k, obtaining stronger asymptotic estimates. We finally provide an explicit construction to obtain an upper bound whose leading coefficient matches that of the lower bound up to a constant multiple of $1/m^3$.

The results in this manuscript highlight a qualitative difference between the unrestricted and the symmetric settings. In the general rectilinear case, $\lim_{n\to\infty} \overline{\mathrm{cr}}(K_n)/\binom{n}{4}$ is known to lie in the interval $[0.379972, 0.380449]$ (cf. [10, 11, 20]), and determining its limit remains open. By contrast, our main result (Theorem 4) determines the value of this limit for m-fold symmetric point sets when m is even:

$$\lim_{n\to\infty} \frac{\text{sym-}\overline{\mathrm{cr}}_m(K_n)}{\binom{n}{4}} = \frac{1}{2}.$$

For odd $m \ge 5$, our lower and upper bounds (Theorem 5) yield the interval

$$\frac{1}{2} - \frac{m-1}{2m^2} - \frac{97m - 31}{270m^3(m+1)} \le \lim_{n\to\infty} \frac{\text{sym-}\overline{\mathrm{cr}}_m(K_n)}{\binom{n}{4}} \le \frac{1}{2} - \frac{m-1}{2m^2}. \tag{14}$$

(Note that $\lim_{n\to\infty} \text{sym-}\overline{\mathrm{cr}}_m(K_n)/\binom{n}{4}$ exists, the proof is in the appendix.)

In particular, for $m = 5$ and $m = 7$ we respectively obtain

$$0.417758 < \frac{21149}{50625} \le \lim_{n\to\infty} \frac{\text{sym-}\overline{\mathrm{cr}}_5(K_n)}{\binom{n}{4}} \le \frac{21}{50} = 0.42,$$

and

$$0.4379 < \frac{751}{1715} \le \lim_{n\to\infty} \frac{\text{sym-}\overline{\mathrm{cr}}_7(K_n)}{\binom{n}{4}} \le \frac{43}{98} < 0.4388.$$

The intervals in (14) are disjoint for distinct odd m, which implies that if m_1 and m_2 are odd with $m_1 < m_2$, then $\text{sym-}\overline{\mathrm{cr}}_{m_1}(K_n) < \text{sym-}\overline{\mathrm{cr}}_{m_2}(K_n)$ when n is large enough. Moreover, as

$m \to \infty$ the even and odd cases come together: both our lower and upper bounds, normalized by $\binom{n}{4}$, approach $\frac{1}{2}$; that is,

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{\text{sym-}\overline{\text{cr}}_m(K_n)}{\binom{n}{4}} = \frac{1}{2}.$$

The most important open problem left is the following:

▶ **Problem 18** (Sharper bounds for odd m). *For odd $m \geq 5$, narrow the gap between the bounds in* (14)*. In particular, identify the correct leading constant of* sym-$\overline{\text{cr}}_m(K_n)$*, i. e., determine the value of*

$$\lim_{n\to\infty} \frac{\text{sym-}\overline{\text{cr}}_m(K_n)}{\binom{n}{4}}$$

for each fixed odd m.

The lower bound in Inequality (14) still applies to $m = 3$ and it matches the best lower bound of 277/729 for the unrestricted problem $\overline{\text{cr}}(K_n)$. However, the construction in Section 4.2 works only for $m \geq 5$. The good news is that the best upper bound in the general case is based on a duplication strategy [10], where larger sets of points are obtained recursively by placing a duplicate of each point in a very small neighborhood around it. If the initial set of points is 3-fold symmetric, then the construction can be carried out so that it preserves symmetry. We strongly believe that determining $\overline{\text{cr}}(K_n)$ is as hard as finding sym-$\overline{\text{cr}}_3(K_n)$ and support the conjecture in [5], which we can now state as follows:

▶ **Conjecture 19** (3-Fold symmetry). *For every n divisible by 3 there exists a 3-fold symmetric point set attaining $\overline{\text{cr}}(K_n)$; that is, if n is divisible by 3, then $\overline{\text{cr}}(K_n) =$ sym-$\overline{\text{cr}}_3(K_n)$.*

For $m \geq 5$, the duplication strategy from [10] produces more crossings than the construction in Section 4.2. It seems that for $m \geq 5$ an important feature to minimize crossings is the property that we used in Lemma 9: We proved that for each fixed m, and every n divisible by m, there exists an m-fold symmetric point set P attaining sym-$\overline{\text{cr}}_m(K_n)$ such that the convex hull of P is an m-gon and the remaining points lie in the central region determined by the halving edges of the convex hull (this region does not exist when $m = 3$). Although we did not need it, one can show that the convex hull of every optimal construction is a convex m-gon. We conjecture the following stronger recursive property.

▶ **Conjecture 20** (Recursive optimality under orbit removal). *If P attains* sym-$\overline{\text{cr}}_m(K_n)$*, then the convex hull of P is an m-gon and if P' is obtained by removing the m points in the convex hull, then the convex hull of P' is an m-gon and the remaining points lie in the central region determined by the halving edges of the convex hull of P'.*

Finally, our construction for odd m in Section 4.2 depends on the choice of a wing set W. We leave the following problem:

▶ **Problem 21** (A better wing set). *Is there a choice of W that improves the upper bound in Theorem 5? More generally, can every point set achieving $\overline{\text{cr}}(K_n)$ be transformed (flattened and stretched) into one whose x-coordinates grow exponentially?*

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6 Appendix

We prove the following lemma:

▶ **Lemma 22.** *For every integer $m \geq 2$,*

$$\lim_{n\to\infty} \frac{\text{sym-}\overline{\text{cr}}_m(K_n)}{\binom{n}{4}}$$

exists.

Proof. Let $\text{sym-}\overline{\text{cr}}_m{}^*(K_n)$ be the minimum number of crossings $\text{Cr}^*(P)$ among all m-fold symmetric drawings P of K_n where we count only those crossings whose four endpoints lie in distinct orbits. The remaining crossings (i.e., those where at least two endpoints belong to the same orbit) are at most $\binom{n/m}{3}m^4 = O(n^3)$. Hence,

$$0 \leq \text{sym-}\overline{\text{cr}}_m(K_n) - \text{sym-}\overline{\text{cr}}_m{}^*(K_n) \leq O(n^3). \tag{15}$$

Fix a point set P attaining $\text{sym-}\overline{\text{cr}}_m{}^*(K_n)$, and $[x_1], [x_2], \ldots, [x_{n/m}]$ be its orbits. For $1 \leq i \leq n/m$, let $P_i = P \setminus [x_i]$. Because every crossing in P with endpoints in distinct orbits belongs to exactly $n/m - 4$ sets P_i, it follows that

$$\left(\frac{n}{m} - 4\right) \text{sym-}\overline{\text{cr}}_m{}^*(K_n) = \left(\frac{n}{m} - 4\right) \text{Cr}^*(P) = \sum_{i=1}^{n/m} \text{Cr}^*(P_i) \geq \frac{n}{m} \text{sym-}\overline{\text{cr}}_m{}^*(K_{n-m}),$$

and consequently

$$\frac{\text{sym-}\overline{\text{cr}}_m{}^*(K_n)}{\binom{n/m}{4}} \geq \frac{\text{sym-}\overline{\text{cr}}_m{}^*(K_{n-m})}{\binom{n/m-1}{4}},$$

Thus, the sequence $\{\text{sym-}\overline{\text{cr}}_m{}^*(K_{mJ})/\binom{J}{4}\}_{J=1}^{\infty}$ is nondecreasing and bounded above by 1, so its limit exists. By (15), the same holds for $\text{sym-}\overline{\text{cr}}_m(K_n)/\binom{n}{4}$. ◀